

## $G$ -flux in F-theory and algebraic cycles

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### Abstract

We construct explicit  $G_4$  fluxes in F-theory compactifications. Our method relies on identifying algebraic cycles in the Weierstrass equation of elliptic Calabi-Yau fourfolds. We show how to compute the  $D3$ -brane tadpole and the induced chirality indices directly in F-theory. Whenever a weak coupling limit is available, we compare and successfully match our findings to the corresponding results in type IIB string theory. Finally, we present some generalizations of our results which hint at a unified description of the elliptic Calabi-Yau fourfold together with the four-form flux  $G_4$  as a coherent sheaf. In this description the close link between  $G_4$  fluxes and algebraic cycles is manifest.

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## 1 Introduction

In the recent years there has been increasing interest in model building with 7-branes in type IIB/F-theory [1–9]. The main advantage of such constructions is that it is in principle possible to both realize moduli stabilization [10, 11] and construct particle physics models at the same time.

F-theory [12, 13] is the most suitable setup to describe type IIB backgrounds with 7-branes, as it combines both the internal compactification manifold and the axio-dilaton profile into one geometric entity. For a review on F-theory, see for example [14–17].

However, F-theory does not geometrize all the relevant data of a type IIB model with 7-branes. Besides the construction of an elliptically fibered CY fourfold, an important ingredient is the presence of fluxes: Three-form fluxes are needed to stabilize part of the geometric moduli of  $B_3$ , while two-form gauge fluxes on the branes are necessary to generate four dimensional chiral matter at the 7-brane intersections. The latter are also important, as they control the presence of chiral zero-modes localized wherever a Euclidean D3-instanton intersects a matter brane. Such modes are discussed in [18–27]. In F-theory, both kinds of fluxes are encoded in a four-form flux  $G_4$  on  $X_4$  (see also [14, 15]). Both the geometry ( $X_4$ ) and the flux ( $G_4$ ) can be given physical interpretations in M-theory. In the limit of vanishing elliptic fiber, M-theory compactified on  $X_4$  is dual to type IIB compactified on  $B_3$ . This duality can also be used to get information on the low energy effective theory of F-theory compactifications with fluxes: One considers the reduction of the M-theory action [28, 29] and then takes the F-theory limit [11, 30–32].

In the last years, there has been much effort to construct GUT models using F-theory setup. The GUT group is realized by an  $ADE$ -singularity along a surface  $S$  in the base space  $B_3$ . The matter fields reside on curves of  $S$  over which the singularity enhances [33]. Since the gauge dynamics is localized on the surface  $S$ , most of the properties of the gauge theory are already encoded in the neighborhood of  $S$  in the fourfold. Such a local approach has been intensively studied in the last few years. It allowed to achieve many phenomenological requirements, see [1–7, 34–50] for an (incomplete) list of references.

To have 4D chiral fields living on matter curves, one has to introduce a two-form flux  $F_2$  for the gauge theory living on  $S$ . In the fourfold, this kind of flux must of course come from a flux  $G_4$ , requiring an understanding of the whole fourfold  $X_4$  in general. In spite of the success of the local models, it is not always clear whether a local model can be embedded into a globally defined one [39, 51, 52]. In [9, 53–61] such embeddings have been constructed, leading to interesting global realizations of GUT models in F-theory. Note that aspects like the tadpole cancellation conditions, which are relevant for consistent models, can only be addressed in a global model.

The geometry of the fourfold used to realize such global models is fairly well understood. However, the four-form flux needed to have 4D chiral matter (and for moduli stabilization) is much less under control. Following [62, 63], the powerful technique of the spectral cover was introduced in [5, 7] to describe the gauge flux in F-theory, and to compute the number of chiral modes generated by this flux on the matter curves. This technique is mutated from the dual Heterotic String Theory. Basically, this four-form flux is defined using a line bundle on the

spectral cover of the GUT surface  $S$ . In spite of several successful results, it is still not completely clear whether this approach captures all the global aspects of the four-form flux. In fact, this construction is local in the sense that the flux is defined near the surface  $S$ . Only when  $X_4$  is a K3 fibration, so that a Heterotic dual exists, the flux is guaranteed to be extendable to the whole  $X_4$ . In building global models it has been essentially assumed that this can be done also when  $X_4$  does not admit an Heterotic dual. Some tests of this assumption have been successful, e.g. a way to compute the  $D3$ -tadpole has been tested in situations without Heterotic dual in [9]. A proposal to generalize the spectral cover construction of the flux to backgrounds that do not have an Heterotic dual has been given in [64] and refined in [65]. Recently, a global description of  $G_4$  fluxes corresponding to the diagonal  $U(1)$  within a  $U(N)$  GUT group has been given in [66].

In this paper we present a new way to construct  $G_4$  flux in F-theory compactifications related to gauge flux on 7-branes. Our fluxes are globally defined from the start, and do not rely in any way upon the extension of local fluxes. Our technique allows us to explicitly compute the induced  $D3$ -tadpole, and the induced chirality along the intersection of the  $I_1$  locus with some  $SU(N)$  enhancement.

We present a geometric characterization of the four-form flux in terms of its Poincaré dual four-cycle. This has several advantages: The four-cycle related to the flux is defined in an algebraic way, so that the flux is automatically of type  $(2, 2)$ . Of course, the introduction of such a flux will fix some complex structure moduli; we will be explicit on such a stabilization. When we require that the flux generates chiral modes, the restriction in the complex structure moduli space is the same as the one found in [67] to have massless  $U(1)$ s. In addition, the algebraic origin of the flux makes it easy to determine its quantization properties and compute the  $D3$ -tadpole and the number of chiral modes.

The tadpole  $-\frac{1}{2} \int_{X_4} G_4 \wedge G_4$  is directly obtained by computing the self-intersection of the corresponding four-cycle. In case the weak coupling limit is smooth, we are able to map the  $G_4$  flux to a two-form flux on the brane and to verify that they induce the same  $D3$ -charge.

The number of 4D chiral modes on a matter curve is computed by integrating the four-form flux on an appropriate four-cycle. This four-cycle is related to the matter curve in the following way: The matter curve is the locus in  $B_3$  over which there is an enhancement of the singularity. This implies that a new two-cycle, which is fibered over the matter curve, appears in the resolution. This fibration gives a new four-cycle called the matter surface. This was already suggested by the duality with Heterotic String Theory in [3, 6] and was related to the spectral cover computations [64, 65]. In this paper we show how to explicitly identify matter surfaces and compute integrals of the fluxes. In situations where the weak coupling limit is smooth, we compare this number with the one obtained by the index theorem in perturbative type IIB.

We will present several examples. In particular, in our final example, we describe a situation with a non-abelian singularity on a surface  $S$ . We study four-form fluxes related both to type IIB fluxes along the Cartan of the non-abelian gauge group and to fluxes that do not live on the brane  $S$ . The last ones are important because they do not break the GUT group, and can give chirality both to charged matter and to GUT singlets.

## 2 Main idea and summary

As the bulk of this paper is somewhat technical, we summarize our main ideas and results here. To motivate the subsequent discussion, let us start by compiling the crucial properties of cycles that can carry supersymmetric  $G_4$  fluxes. In a supersymmetric minimum of a compactification of F-theory on an elliptic Calabi-Yau fourfold,  $G_4$  flux has to be of type  $(2, 2)$ , primitive [28] and have one leg in the fiber [11]. The last condition has to be satisfied in order for the flux not to break Lorentz symmetry in the four-dimensional effective theory. We can rephrase this condition by demanding that the four-form  $G_4$  should integrate to zero on any divisor of the base or the elliptic fibration over a curve in the base. Hence the flux cannot be simply the Poincaré dual of a complete intersection of the Weierstrass equation with two divisors of the ambient space.

The condition that the flux should be of type  $(2, 2)$  can be easily satisfied if we describe the four-cycle which is Poincaré dual to  $G_4$  by algebraic equations. We are hence interested in situations in which fourfolds described by a Weierstrass equation gain extra algebraic cycles. To investigate under which conditions we can find such extra cycles we rewrite the Weierstrass model in a specific form, as is discussed below.

We start with the Tate form of the elliptic fibration:

$$Y^2 + a_1XYZ + a_3YZ^3 = X^3 + a_2X^2Z^2 + a_4XZ^4 + a_6Z^6. \quad (2.1)$$

One can shift  $X$  and  $Y$  to bring this into the Weierstrass form

$$y^2 = x^3 + xz^4f + z^6g. \quad (2.2)$$

The shift of coordinates taking us from the Weierstrass to the Tate form yields  $f$  and  $g$  as functions of the  $a_i$ , see appendix A. We will use this parametrization in the following, as it allows us to use the simpler Weierstrass form while still having all the information contained in (2.1). This parametrization has the further advantage of being equivalent to the parametrization used in Sen's weak coupling limit [68, 69].

Our construction of fluxes starts from using the parametrization of  $f$  and  $g$  in terms of the sections  $a_i$  appearing in the Tate form and rewriting the Weierstrass model as

$$Y_-Y_+ - z^6a_6 = XQ. \quad (2.3)$$

Here we have defined the quantities

$$\begin{aligned} Y_{\pm} &= y \pm \frac{1}{2}z^3a_3 \\ X &= x - \frac{1}{12}z^2b_2 \\ Q &= (x - \frac{1}{12}z^2b_2)(x + \frac{1}{6}z^2b_2) + \frac{1}{2}z^4b_4 \\ &= X(X + \frac{1}{4}z^2b_2) + \frac{1}{2}z^4b_4. \end{aligned} \quad (2.4)$$

We have collected the details of this reformulation in appendix A.

For our cases of interest, (2.3) defines an elliptic Calabi-Yau fourfold as a hypersurface in a 5-dimensional ambient space. When  $a_6$  factorizes in a non-trivial way,  $a_6 \equiv \rho\tau$ , we find that the fourfold gains extra algebraic  $(2, 2)$  cycles. These cannot be written as complete intersections of the Weierstrass equation with divisors of the ambient space, but take for instance the form

$$\sigma_{\rho} : \quad \{Y_- = 0\} \quad \cap \quad \{X = 0\} \quad \cap \quad \{\rho = 0\} \quad (2.5)$$

in the ambient space. We may then construct a four-cycle which has one leg in the fiber, i.e. is orthogonal to horizontal and vertical divisors, by defining

$$\gamma_\rho \equiv \sigma_\rho - [\rho] \cdot F. \quad (2.6)$$

Here,  $F$  is the hyperplane section of the  $\mathbb{P}_{123}^2$  the elliptic fiber is embedded in. It is chosen such that  $x$  and  $y$  are sections of  $F^2$  and  $F^3$ , respectively. Constructions of this type can also be performed in many situations in which the polynomials  $a_i$  already factorize such that we have non-abelian gauge groups. Due to the form (2.6), the flux we have constructed cannot be represented as the wedge product of two two-forms inside  $X_4$ .

We claim that putting a flux  $G_4 = \gamma_\rho$  corresponds to having a supersymmetric flux  $F_2$  in the gauge theory on the 7-branes. Note that  $\gamma_\rho$  is defined using a divisor class  $[\rho]$  of the base, which intersects the 7-branes in curves. The 7-brane divisors have the form

$$\eta^2 - \xi^2 (\psi^2 - \rho \tau) = 0 \quad (2.7)$$

where  $\eta$  is a polynomial, and  $\xi \rightarrow -\xi$  and the orientifold involution. Then, these curves are given by  $\rho = 0 \cap \eta = \pm \xi \psi$ . They are Poincaré dual to two-forms of type  $(1, 1)$ , so that it seems natural to identify

$$G_4 = \sigma_\rho - [\rho] \cdot F \quad \longleftrightarrow \quad F_2 = PD(\{\rho = 0 \cap \eta = \pm \xi \psi\}). \quad (2.8)$$

Using our explicit realization of  $G_4$ , we can compute the  $D3$  tadpole by integrating

$$-\frac{1}{2} \int_{CY_4} G_4 \wedge G_4. \quad (2.9)$$

Using our proposed relation to the gauge flux on the  $D7$ -branes, we compute its tadpole also from the type IIB perspective. We compare the results whenever the weak coupling limit is available, finding complete agreement.

A gauge flux  $F_2$  can also lead to chirality for charged matter. In F-theory, charged matter can reside at the intersections of branes. Over such matter curves, there are extra vanishing spheres which encode the representations of the matter multiplets. Via small resolutions of these singularities to make those spheres finite, one naturally produces a four-cycle  $\hat{C}$  which is a collection of spheres fibered over the matter curve. The conservation of induced  $D3$ -charge implies that, upon resolving the singularity, our  $G_4$  flux in (2.8) will itself undergo a transition to a new flux  $\tilde{G}_4$ . It seems natural [3, 6] that the chirality induced by this new flux can be computed by integrating

$$\int_{\hat{C}} \tilde{G}_4. \quad (2.10)$$

Again, we use the identification (2.8) and find agreement between the results obtained in F-theory in the weak coupling limit and type IIB for all configurations we consider.

In models with non-abelian gauge groups, one can get four-cycles of type  $(2, 2)$  by appropriately resolving the singularities. These four-cycles are obtained by fibering the exceptional curves of the resolution over a divisor of the non-abelian brane stack. Fluxes constructed in this way correspond to  $F_2$  in the Cartan of the non-abelian gauge group. Even though they can generate a chiral spectrum, they also necessarily break the gauge group to the commutant subgroup.

Our construction is fundamentally different in this regard: The flux (2.6) induces chirality but does not break the non-abelian gauge group. Moreover, because we can build a global flux over the  $I_1$  locus of the discriminant, it can generate chiral modes for singlets of the non-abelian (GUT) group.

On the type IIB side, we make extensive use of the technique of  $D7$ -brane ‘deconstruction’, introduced in [70]. Besides allowing us to treat situations with a singular  $D7$ -brane locus in an elegant way, the main conceptual advantage of this description is the unification of the  $D7$ -branes and their gauge bundles into a single object. From a mathematical point of view, this means that we are describing the  $D7$ -brane as a coherent sheaf  $\mathcal{E}$  defined by the short exact sequence

$$0 \rightarrow E \xrightarrow{T} F \rightarrow \mathcal{E} \rightarrow 0. \quad (2.11)$$

Here,  $E$  ( $F$ ) is a vector bundle on a stack of  $D9$  ( $\overline{D9}$ )-branes and  $T$  is the tachyon that makes this unstable configuration collapse into a  $D7$ -brane. Consequently, the locus of the  $D7$ -brane is given as the determinant of the tachyon matrix. The flux, which is determined by the bundles  $E$  and  $F$ , has a natural algebraic description in this setting.

This work indicates that a similar structure exists in F-theory compactifications on elliptic Calabi-Yau fourfolds. We can write down an exact sequence which gives the Calabi-Yau fourfold together with the  $G_4$  flux as a coherent sheaf. The role of the determinant of the tachyon matrix is played by the Weierstrass equation. It turns out that the latter can be written as the Pfaffian of an anti-symmetric matrix. Hence its rank goes down by multiples of two (instead of one in the IIB case) so that we end up with a vector bundle on the elliptic Calabi-Yau fourfold. The desired  $G_4$  flux can then be constructed from the second Chern class of this vector bundle.

This paper is organized as follows. In section 3 we explain our ideas in the case of elliptic K3 surfaces, which already captures some of the salient ingredients of our construction. In this case, we aim for non-singular Weierstrass models for which further integral cycles of type  $(1, 1)$  are present.

In section 4 we come to elliptic fourfolds, which are our main object of interest. After explaining the construction of fluxes in this case, we compute the  $D3$  tadpole and the number of chiral modes in various models. In particular, we consider the following configurations: We start with a single 7-brane, so that the elliptic fourfold is smooth. We then describe a fourfold for which  $a_6 \equiv 0$ . In this case, which has been dubbed  $U(1)$  restricted Tate model in [67], there is an intersection of the 7-brane with itself. This leads to matter localized at the intersection which becomes chiral for non-zero fluxes. After this, we turn to models with non-abelian gauge symmetries. As toy models for F-theory GUTs, we consider configurations with ‘non-split’  $SU(2)$  ( $Sp(1)$ ) and ‘split’  $SU(2)$  singularities. In these cases, flux of the type (2.6) induces chirality in the case of  $U(1)$  restricted Tate models.

In section 5 we study the weak coupling limits of the configurations considered in section 4. We compute the  $D3$  tadpoles and the chiral indices from the type IIB perspective using the methods developed in [70] and find complete agreement with the results obtained in F-theory.

Finally, in section 6, we comment on a description of elliptic fourfolds together with  $G_4$  fluxes in terms of coherent sheaves, and find an interesting link to the techniques of matrix factorizations. To show the power of this description, we reproduce the  $D3$ -tadpole computation in a very quick way, via a simple application of the Grothendieck-Riemann-Roch formula for the push-forward of a rank two vector bundle.



### 3 Algebraic cycles in the K3 surface

In this section we develop our ideas in the simplified setting of K3. For compactifications of F-theory on  $K3 \times K3$ , one of the two K3 surfaces has to be elliptic. In this model, all branes are points in the elliptic K3 and wrap the other K3 completely. We are interested in a  $G_4$  flux of the form

$$G_4 = F_2 \wedge \gamma, \quad (3.1)$$

where  $F_2$  is a two-form flux on a brane.

As supersymmetry demands that  $G_4$  is of type  $(2, 2)$  and  $F_2$  is of type  $(1, 1)$ ,  $\gamma$  must be an element of the Picard group of the elliptic K3:

$$\text{Pic}(X) \equiv H^{1,1}(X) \cap H^2(X, \mathbb{Z}). \quad (3.2)$$

It is well-known that the rank of the Picard group, which is called the Picard number, is zero for generic K3 surfaces. Even though there are 20 independent harmonic  $(1, 1)$  forms for any K3, none of them is in the *integral* cohomology generically. This is not the case if the K3 surface in question is embedded in a projective space or toric variety: intersections with divisors of the ambient space will descend to non-trivial elements of the Picard group. For a given hypersurface or complete intersection, the rank of the Picard group can be further enhanced by appropriately choosing a non-generic form for its defining equations. This will be the crucial ingredient in our construction.

For an elliptic K3 surface described by a Weierstrass model, the two independent toric divisors of the ambient space give rise to a two-dimensional Picard group. Its generators can be chosen to be the base and the fiber of the elliptic fibration. Hence the dual two-forms will either have two or no legs along the fiber directions. Hence we break Lorentz symmetry after taking the F-theory limit if we use one of those forms for  $\gamma$ . Thus, we are interested in finding K3 surfaces which have further integral  $(1, 1)$  cycles. We now discuss those enhancements of the Picard group as well as their physical interpretation.

Generically, the elliptic fiber degenerates over 24 points in the base  $\mathbb{P}^1$  of the elliptically fibered K3. If some of these points come together, this worsens the singularity of the elliptic fiber and leads to ADE singularities of the K3 surface. From the physical point of view, these singularities signal the appearance of non-abelian gauge symmetries due to coincident branes. The vanishing cycles of these singularities precisely correspond to the Cartan generators of the gauge groups. One can then enhance the Picard group by blowing up these singularities. Physically, this means moving onto the Coulomb branch of the gauge theory. The exceptional cycles are obviously integral and of type  $(1, 1)$ . Furthermore, their intersection numbers with the base and the fiber of the elliptic fibration are vanishing. Hence these cycles can be used to construct  $G_4$  flux corresponding to  $F_2$  along the elements of the Cartan subalgebra.

The process of blowing up does not physically separate the branes in F-theory: Taking the F-theory limit (i.e. vanishing fiber size), the exceptional curves are blown down again [71]. From the perspective of type IIB, it is clear that we can have a supersymmetric two-form flux  $F_2$  also along the  $U(1)$ s of separate  $D7$ -branes, without having any non-abelian gauge symmetries, however. Hence this situation is fundamentally different from the one considered above.

We now discuss how (2.3) can gain extra algebraic cycles without gauge enhancement and



blow-ups. The crucial observation is that the Weierstrass model is the sum of two factors,

$$Y_- Y_+ = XQ. \quad (3.3)$$

if  $a_6 \equiv 0$ , i.e. when  $b_6$  becomes a square. This means we restrict the complex structure moduli of K3 in a certain way. One can check that the locus where  $a_6 \equiv 0$  is at codimension one in moduli space.

The curves  $\sigma_{\pm}$  defined by taking the two equations

$$Y_{\pm} = 0 \quad \text{and} \quad X = 0, \quad (3.4)$$

in the ambient space are automatically inside K3 for the configuration we consider. As we can explicitly represent these curves by algebraic equations, it follows that the corresponding cycles are of type  $(1, 1)$  and integral. Note that these curves cannot be written as complete intersections of some divisor with the K3 surface, so that they are not equivalent to some linear combinations of the base and the fiber of the elliptic fibration. The K3 hypersurface generically remains smooth for  $a_6 \equiv 0$  because a singularity only occurs if all four factors in the ambient threefold vanish simultaneously.

If we move away from the locus in moduli space where  $a_6 \equiv 0$ , the integral cycle we have constructed ceases to be of type  $(1, 1)$ . Consequently, we lose the description (3.4), as these curves are not inside K3 for  $a_6 \neq 0$ .

Let us now study some of the properties of these curves. First note that they do not intersect the standard section which sits at  $z = 0$ ,  $y^2 = x^3$ : Setting  $z = 0$  forces  $y = x = 0$  which is part of the exceptional set of the ambient toric variety. Second, (3.4) determines a single point in the fiber for every point on the base, so that it intersects the fiber cycle exactly once. Hence the cycles  $\sigma_{\pm}$  have all the properties one expects from a section of the elliptic fibration. As they are holomorphic sections of the elliptic fibrations,  $\sigma_+$  and  $\sigma_-$  both have the same topology as the base, i.e. each is a  $\mathbb{P}^1$ . Furthermore, as  $\sigma_{\pm}$  are curves in K3, their self-intersection numbers must be equal to  $-2$ .

Elliptic fibrations of the form (3.3), which allow for a second section, have been studied in [72–77]. In particular, they are related to fibrations of  $E_7$  type, for which the elliptic fibre is embedded in  $\mathbb{P}_{1,1,2}^2$  instead of  $\mathbb{P}_{2,3,1}^2$ . This has been discussed in the context of elliptic fourfolds in [67] recently.

We can now easily construct integral cycles of type  $(1, 1)$  which are orthogonal to base  $\beta$  and fiber  $\phi$  by defining

$$\gamma_{\pm} = \sigma_{\pm} - \beta - 2\phi, \quad (3.5)$$

so that<sup>1</sup>

$$\gamma_{\pm} \cdot \phi = 0, \quad \gamma_{\pm} \cdot \beta = 0. \quad (3.6)$$

Using (3.5), one can also compute that  $\gamma_{\pm} \cdot \gamma_{\pm} = -4$ . As there cannot be any connected algebraic curves inside K3 which have a self-intersection number smaller than  $-2$ , the last equation shows that the cycles  $\gamma_{\pm}$  can only be represented by the disjoint union of at least two curves.

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<sup>1</sup>Remember that base and fiber have the following intersection numbers:  $\beta^2 = -2$ ,  $\phi^2 = 0$ ,  $\beta \cdot \phi = 1$ .

The observation of the last paragraph is connected with a physical interpretation of the constraint  $a_6 \equiv 0$ . By going to the weak coupling limit, such configurations are realized when two pairs of  $D7$ -branes align such that their displacements in  $\mathbb{P}^1$  are pairwise equal. As these displacements are measured by periods (see e.g. [78, 79]) of the holomorphic two-form of K3, we can find two pairs of integral cycles  $\gamma_1, \gamma'_1$  and  $\gamma_2, \gamma'_2$  for which

$$\int_{\gamma_1} \Omega^{2,0} = \int_{\gamma'_1} \Omega^{2,0}, \quad \int_{\gamma_2} \Omega^{2,0} = \int_{\gamma'_2} \Omega^{2,0}. \quad (3.7)$$

The cycles  $\gamma_1 - \gamma'_1$  and  $\gamma_2 - \gamma'_2$  are hence orthogonal to  $\Omega$ , so that they are integral cycles of type  $(1, 1)$ . We can identify these cycles with the curves  $\gamma_{\pm}$  we have found from the Weierstrass model. Indeed one can check that also their intersection numbers agree. Furthermore, we only have to adjust the position of a single  $D7$ -brane to get to the desired configuration. Hence the extra integral cycles of type  $(1, 1)$  also appear at codimension one in complex structure moduli space from this point of view.

## 4 Algebraic cycles and fluxes for fourfolds

Before describing the analogue of the algebraic cycles we have constructed for K3 in the last section, let us make some definitions to fix our notation. Let  $B_3$  be the base of  $X_4$ ,  $K$  its canonical and  $\bar{K}$  its anti-canonical bundle. The first Chern class of the base is  $c_1(B_3) = \bar{K}$ . We are using the same symbol for a line bundle, its first Chern class, and its associated divisor. We consider the Weierstrass model (2.3) as a hypersurface in an ambient space  $X_5$  which is the total space of the weighted projective bundle:

$$X_5 \equiv \mathbb{P}_{2,3,1}(\mathcal{O}_{B_3} \oplus \mathcal{O}_{B_3} \oplus K). \quad (4.1)$$

In other words, we define the three coordinates  $(x, y, z)$  to be sections of  $\mathcal{O}_{B_3}, \mathcal{O}_{B_3}, K$ , respectively, and then quotient the space by a  $\mathbb{C}^*$  action with the weights  $(2, 3, 1)$ . Let  $F$  denote the divisor class corresponding to the projective action in the fiber directions, so that  $x, y, z$  are sections of  $F^{\otimes 2}, F^{\otimes 3}, F \otimes K$ , respectively. This is summarized in table 1

$x$	$y$	$z$
0	0	$K$
2	3	1

Table 1: The rows indicate the projective weights of the coordinates under the toric  $\mathbb{C}^*$  action on the fiber coordinates of the  $X_5$  space, and their transformation properties w.r.t. transitions along the base.  $K$  denotes the canonical bundle of  $B_3$ .

The Stanley-Reisner ideal of  $X_5$  is simply given by the element  $xyz$ . From it, we deduce the relation  $F^3 = F^2 \bar{K}$ .

Furthermore, we have that

$$6F^2 = PD(B_3 \subset X_5). \quad (4.2)$$

This corresponds to the fact that fixing the values of  $x$  and  $y$  gives us a copy of  $B_3$  in  $X_5$ .

As before, we write the Weierstrass equation in the form

$$Y_- Y_+ - z^6 a_6 - XQ = 0. \quad (4.3)$$

## 4.1 Construction of the flux

We now generalize the construction of the last section to the case of elliptically fibered Calabi-Yau fourfolds. In this case, the form (3.3) for the Weierstrass model would lead to a conifold singularity fibered over a curve in the base. We can, however, extend the structure used in the last section by demanding  $a_6$  to factorize in a non-trivial way:  $a_6 \equiv \rho\tau$ . In this case, the Weierstrass equation for the elliptic fourfold  $X_4$  can be written as

$$Y_-Y_+ - z^6\rho\tau - XQ = 0. \quad (4.4)$$

As before, this equation generically describes a smooth fourfold: In order for the equation and its gradients to vanish one would have to solve 6 equations in the ambient fivefold, which generically is not possible for dimensional reasons.

We can now find an algebraic four-cycle which is automatically inside  $X_4$  by imposing three equations in the ambient fivefold. Let us consider the following<sup>2</sup>:

$$\sigma_\rho : \{Y_- = 0\} \cap \{X = 0\} \cap \{\rho = 0\} \subset X_5. \quad (4.5)$$

The first two equations eliminate the coordinates  $y$  and  $x$ . In addition, they forbid  $z$  from vanishing, since  $xyz$  is in the SR-ideal. Hence, the first two equations define a threefold that is isomorphic to  $B_3$  (but not contained in  $X_4$ ). Therefore,  $\sigma_\rho$  is a ‘horizontal’ surface that is isomorphic to the hypersurface  $\rho = 0 \subset B_3$ . Note that  $\sigma_\rho$  does not intersect the ‘old’ section of the fibration given by  $z = 0$ , and hence none of the ‘horizontal’ surfaces. As  $\sigma_\rho$  is a section of the elliptic fibration over  $\rho = 0 \subset B_3$ , it intersects each fiber and hence also four-cycles given by the intersection of two equations in  $B_3$ .

We now construct the fourfold analogue of (3.5). Our goal is to create a cycle  $\gamma$  which satisfies

$$\gamma \cdot D_i \cdot D_j = 0, \quad \forall i, j \quad \text{and} \quad \gamma \cdot D_i \cdot (F + K) = 0, \quad (4.6)$$

for any divisor  $D_i$  of the base. Note that  $F + K$  is the class corresponding to the section at  $z = 0$  of the Weierstrass model. As  $K$  is also a divisor of the base, these equations imply that  $\gamma \cdot D_i \cdot F = 0$ .

As  $\sigma$  does not meet the section at  $z = 0$ , we find that  $\sigma_\rho \cdot D_i \cdot (F + K) = 0$ . However, using the fact that  $\sigma$  meets every fiber over  $B_3 \cap [\rho]$  in a single point, one can deduce that

$$\int_{X_4} \sigma_\rho \cdot D_i \cdot D_j = \int_{B_3} [\rho] \cdot D_i \cdot D_j, \quad (4.7)$$

where  $[\rho]$  is the Poincaré dual to  $\rho = 0$ . In order to eliminate this intersection, one immediate remedy is to define the cycle  $\sigma_\rho - [\rho] \cdot F$ , since

$$\int_{X_4} F \cdot [\rho] \cdot D_i \cdot D_j, = \int_{X_4} (F + K) \cdot [\rho] \cdot D_i \cdot D_j, \quad (4.8)$$

$$= \int_{B_3} [\rho] \cdot D_i \cdot D_j. \quad (4.9)$$

---

<sup>2</sup>We can also pick the inequivalent branch with  $Y_+ = 0$ . The sum of the two branches  $Y_-$  and  $Y_+$  is a complete intersection of the Weierstrass model with  $X = 0$  and  $\rho = 0$ .

The first equality follows from the fact that  $K, \rho, D_i, D_j$  are all elliptic fibrations over divisors in the base, so that the intersection between all four of them vanishes.

Since the relation  $F \cdot (F + K) = 0$  holds on  $X_4$ , this new term will not generate any unwanted intersections with ‘horizontal’ four-cycles. Therefore, our coveted<sup>3</sup> four-cycle is given by

$$\gamma_\rho \equiv \sigma_\rho - [\rho] \cdot F. \quad (4.10)$$

If we evaluate this expressions in the ambient space  $X_5$ , we find that

$$\begin{aligned} \gamma_\rho &= 3F \cdot 2F \cdot [\rho] - [\rho] \cdot F \cdot PD(X_4 \subset X_5) \\ &= 6[\rho] \cdot F \cdot F - [\rho] \cdot F \cdot 6F = 0. \end{aligned} \quad (4.11)$$

Hence  $\gamma_\rho$  is a trivial cycle in  $X_5$ . This does, however, not imply that it is also trivial inside  $X_4$ . The cycle  $\sigma_\rho$  cannot be written as the intersection of two divisors inside  $X_4$ , whereas the part we subtract has this form. Hence the two terms can never cancel inside  $X_4$ . In particular,  $\gamma_\rho$  cannot be written as the wedge product of two two-forms of  $X_4$ . As explained in appendix B, triviality in the ambient space  $X_5$  ensures that our flux is orthogonal to all divisors in  $X_4$  which descend from  $X_5$ .

Note that choosing  $\tau$  to be a constant, which corresponds to going back to choosing a completely generic  $a_6$ , means that  $\sigma_\rho$  is an intersection of divisors in  $X_4$ :

$$\sigma_\rho = \{Y_- = 0\} \cap \{X = 0\} \cap \{W = 0\} = 3F \cdot 2F \subset X_4. \quad (4.12)$$

Furthermore,  $[\rho] = 6\bar{K}$  in this case. Hence we find  $\gamma_\rho = 6F \cdot F - 6\bar{K} \cdot F$ . As we have seen before,  $F^2 - \bar{K}F$  vanishes, as it corresponds to an element of the Stanley-Reisner ideal of  $X_4$ , so that  $\gamma_\rho$  is trivial in homology.

Following the same steps as for  $\gamma_\rho$ , we can define another flux

$$\gamma_\tau \equiv \sigma_\tau - [\tau] \cdot F. \quad (4.13)$$

As  $a_6 = \rho\tau = 0$  is a complete intersection of the Weierstrass equation with  $Y_+ = 0$ ,  $X = 0$ , we find that

$$G_\rho + G_\tau = 0, \quad (4.14)$$

so that the four-cycle  $\gamma_\tau$  related to  $\tau$  is equal to minus the four-cycle  $\gamma_\rho$ .

The main difference between divisors in K3 and in Calabi-Yau threefolds is that, in the former, it depends on the complex structure whether an integral two-cycle is of type  $(1, 1)$ . A similar situation arises in  $H^4(X_4, \mathbb{Z})$ . For Calabi-Yau fourfolds, the space  $H^{2,2}$  splits into the primary horizontal and the primary vertical subspace,  $H^{2,2} = H_H^{2,2} \oplus H_V^{2,2}$  [80]. Similar to the case of Calabi-Yau threefolds, derivatives of  $\Omega^{4,0}$  span the whole of  $H^{3,1}$ . Second derivatives of  $\Omega^{4,0}$ , however, fail to reach all of  $H^{2,2}$  [81], but only map to the primary horizontal subspace  $H_H^{2,2} \subset H^{2,2}$ . Furthermore, the primary vertical subspace  $H_V$  is spanned by products of elements of  $H^{1,1}$  [80].

---

<sup>3</sup>Performing the same construction with a cycle that is given by a divisor  $D$  in the base intersected with the section  $z = 0$  yields the four-cycle  $(F + K) \cdot D - D \cdot F = K \cdot D$ . This four-cycle clearly is again an elliptic fibration over a curve in the base, so that it has two legs along the fiber. Such a state of affairs can be expected from the fact that (4.7) does not hold in this case.

The cycle we have constructed, (4.5), cannot be written as the wedge of two two-forms, so that it must belong to  $H_H$ . This also clarifies why we have to fix some of the complex structure moduli: the lattice

$$H_H^{2,2}(X_4) \cap H^4(X_4, \mathbb{Z}), \quad (4.15)$$

which plays a similar role as the Picard lattice of K3, does not contain any elements generically<sup>4</sup>. When we fix some of the complex structure moduli, some integral four-forms can become of type  $(2, 2)$ . This is precisely what happens for the integral cycle (4.10), which becomes of type  $(2, 2)$  when  $a_6$  factorizes as  $a_6 = \rho\tau$ .

#### 4.1.1 The D3-brane tadpole

Turning on a flux  $G_4 = \gamma_\rho$  will induce some  $D3$ -charge equal to  $-\frac{1}{2} \int G_4 \wedge G_4$ . Using our explicit description of  $G_4$ , we will now show how to compute the tadpole in terms of intersections of divisors in the base. In the weak coupling limit, these expressions are reproduced in the corresponding type IIB computation. For the sake of brevity, we shall drop the index  $\rho$  from the cycles  $\gamma$  and  $\sigma$  in this section.

Let us therefore proceed to compute the self-intersection number of  $\gamma$ . As  $\gamma \cdot D_i \cdot F = 0$ , we have

$$\int_{X_4} \gamma^2 = \int_{X_4} \gamma \cdot \sigma = \int_{X_4} \sigma^2 - \sigma \cdot F \cdot [\rho]. \quad (4.16)$$

We may now use that  $\sigma$  defines a section of the elliptic fibration over the locus  $\rho = 0$  to pull the second term down to the base. Hence we find

$$\int_{X_4} \gamma^2 = \int_{X_4} \sigma^2 + \int_{B_3} K \cdot [\rho]^2. \quad (4.17)$$

Since  $\sigma$  is not a complete intersection of two divisors within  $X_4$ , we must use rather indirect techniques in order to calculate  $\sigma^2$ . We first note that  $\sigma$  can be thought of as the zero locus of a section of some rank-two vector bundle  $E$  over  $X_4$  which restricts to the normal bundle of  $\sigma$  as follows:

$$E|_\sigma = N_{\sigma \subset X_4}. \quad (4.18)$$

Then, we can express the self-intersection of  $\sigma$  in terms of the second Chern class of the normal bundle as follows:

$$\sigma \cdot \sigma = \int_\sigma \sigma = \int_\sigma c_2(N_\sigma). \quad (4.19)$$

Since the surface is not given by two equations in  $X_4$ , some work is required to compute this Chern class. The shortest route is to use the following exact sequence of normal bundles:

$$0 \rightarrow N_{\sigma \subset X_4} \rightarrow N_{\sigma \subset X_5} \rightarrow N_{X_4 \subset X_5} \rightarrow 0. \quad (4.20)$$

We know that  $\sigma$  is given by three equations in  $X_5$ , as defined in (4.5). Hence, we can express the middle bundle as follows:

$$N_{\sigma \subset X_5} = F^{\otimes 2} \oplus F^{\otimes 3} \oplus [\rho]. \quad (4.21)$$

---

<sup>4</sup>In contrast,  $H_V^{2,2}(X_4)$  does not depend on the complex structure moduli and one can find a basis of integral cycles.

Similarly, we can express the right-most bundle as

$$N_{X_4 \subset X_5} = F^{\otimes 6}. \quad (4.22)$$

The exact sequence tells us that

$$\begin{aligned} c(N_{\sigma \subset X_4}) &= c(N_{\sigma \subset X_5})/c(N_{X_4 \subset X_5}) \\ &= 1 + ([\rho] - F) + F \cdot (12F - [\rho]). \end{aligned} \quad (4.23)$$

hence,

$$c_2(N_{\sigma \subset X_4}) = F \cdot (12F - [\rho]). \quad (4.24)$$

Integrating this over  $\sigma$  gives us

$$\int_{X_4} \sigma \cdot \sigma = \int_{X_5} 6F^2 \cdot [\rho] \cdot F \cdot (12F - [\rho]) = \int_{B_3} \bar{K} \cdot [\rho] \cdot (12\bar{K} - [\rho]). \quad (4.25)$$

Therefore, the self-intersection of  $\gamma$  is

$$\gamma \cdot \gamma = 2 \int_{B_3} \bar{K} \cdot [\rho] \cdot (6\bar{K} - [\rho]). \quad (4.26)$$

If we turn on a flux  $G_4$  that is Poincaré dual to  $\gamma$ , this will hence induce a  $D3$ -charge given by:

$$Q_{D3}^F = -\frac{1}{2} \int_{X_4} G_4 \wedge G_4 = - \int_{B_3} c_1(B_3) \cdot [\rho] \cdot (6c_1(B_3) - [\rho]) = - \int_{B_3} c_1(B_3) \cdot [\rho] \cdot [\tau], \quad (4.27)$$

where we used  $\bar{K} = c_1(B_3)$ . We will confirm this result from the corresponding type IIB computation in the weak coupling limit in section 5.1.

## 4.2 Recombination and flux change

When a smooth brane supporting two-form flux splits into two (or more) intersecting pieces, the flux quanta change [70]. This is to be expected from the perspective of F-theory: As the Euler numbers of the (resolved) corresponding fourfolds change, the geometric part of the  $D3$ -brane tadpole must also change in this process. This change is compensated by a change in the four-form flux  $G_4$ , which is connected to the flux on the brane world-volume.

Let us discuss this in the present context. We start with a smooth fourfold (4.4), so that there is one smooth recombined 7-brane, and a four-form flux  $G_\rho$ . If we set  $a_6 \equiv 0$  in (4.4), the CY fourfold becomes:

$$Y_- Y_+ = XQ. \quad (4.28)$$

This fourfold is clearly singular over the curve given by  $Y_+ = Y_- = X = Q = 0$ . As the singularity occurs at codimension two in the base, it must be due to intersections between branes. The discriminant locus, however, is given by one connect surface. The intersection of this brane with itself leads to matter charged under its  $U(1)$ . The connection of configurations with  $a_6 \equiv 0$  and an extra  $U(1)$  was discussed in [67], (see also [72–77]) where the corresponding elliptic fibration was called a ‘ $U(1)$  restricted Tate model’. We will also use this terminology in

the following. In the weak coupling limit this configuration corresponds to splitting the otherwise connected  $D7$ -brane into a brane and its orientifold image in the  $CY_3$  double cover.

One can compute the geometric tadpole of  $X_4$  by appropriately resolving its singularities, which we will do in section 4.2.1. The result differs from the geometric tadpole of the smooth fourfold we have started with. In the present situation, the difference is given by (see also [67]):

$$9 \int_{B_3} \bar{K}^3. \quad (4.29)$$

As the tadpole has to be canceled both before and after the transition, the flux must also change such that the total tadpole remains invariant. If we denote the flux in the situation with intersecting branes by  $\tilde{G}_\rho$ , we have that

$$\frac{\chi(X_4)}{24} - \frac{1}{2} \int_{X_4} G_\rho \wedge G_\rho = \frac{\chi(\tilde{X}_4)}{24} - \frac{1}{2} \int_{\tilde{X}_4} \tilde{G}_\rho \wedge \tilde{G}_\rho. \quad (4.30)$$

As

$$\frac{\chi(X_4)}{24} - \frac{\chi(\tilde{X}_4)}{24} = 9 \int_{B_3} \bar{K}^3, \quad (4.31)$$

it must be that

$$-\frac{1}{2} \int_{\tilde{X}_4} \tilde{G}_\rho \wedge \tilde{G}_\rho + \int_{B_3} \bar{K} \cdot [\rho] \cdot (6\bar{K} - [\rho]) = +9 \int_{B_3} \bar{K}^3. \quad (4.32)$$

Hence the tadpole contribution of the remaining flux on the resolved fourfold  $\tilde{X}_4$  can be written as

$$\tilde{Q}_\rho = \int_{B_3} \bar{K} \cdot ([\rho] - 3\bar{K}) \cdot ([\rho] - 3\bar{K}). \quad (4.33)$$

#### 4.2.1 A small resolution

Let us now explicitly resolve the singularity discussed in the previous segment. The singularity of the hypersurface (4.28) has the structure of a conifold singularity fibered over the curve

$$C : \{Y_- = 0\} \cap \{Y_+ = 0\} \cap \{X = 0\} \cap \{Q = 0\} \subset X_5. \quad (4.34)$$

Going back to the definitions of these quantities in (2.4), one sees that this curve can be equivalently described by

$$C : \{y = 0\} \cap \{a_3 = 0\} \cap \{X = 0\} \cap \{b_4 = 0\} \subset X_5. \quad (4.35)$$

In order to define the flux and perform a computation of the induced  $D3$ -tadpole, we need to appropriately resolve this singularity. We could do this via a blow-up, however, it is more natural to perform a small resolution by tagging a  $\mathbb{P}^1$  with homogeneous coordinates  $[\lambda_1 : \lambda_2]$  to the ambient fivefold, resulting in an ambient sixfold  $X_6$  described in table 2, and imposing

$$\tilde{X}_4 : Y_- \lambda_2 = Q \lambda_1 \cap Y_+ \lambda_1 = X \lambda_2 \subset X_6. \quad (4.36)$$



In [82], the necessity for small resolutions in F-theory was stressed. Hence our resolved fourfold  $\tilde{X}_4$  is given by two equations in an ambient sixfold  $X_6$ . The divisor classes of the different homogeneous coordinate used in the construction of  $X_6$  are

$$\begin{aligned} [Y_{\pm}] &= 3F & [X] &= 2F & [z] &= F + K \\ [\lambda_1] &= H & [\lambda_2] &= F + H. \end{aligned} \quad (4.37)$$

We have collected this information in table 2.

$x$	$y$	$z$	$\lambda_1$	$\lambda_2$
0	0	$K$	0	0
2	3	1	1	0
0	0	0	1	1

Table 2: The rows indicate the projective weights of the coordinates under the two toric  $\mathbb{C}^*$  actions for the  $X_6$  space, and the transitions functions w.r.t. patches on the base.  $K$  denotes the canonical bundle of  $B_3$ .

The exceptional sets and the SR-ideal are given by

$$\begin{aligned} \{y = x = z = 0\} &\leftrightarrow F^2 \cdot (F + K) = 0 \\ \{\lambda_1 = \lambda_2 = 0\} &\leftrightarrow H \cdot (F + H) = 0. \end{aligned} \quad (4.38)$$

The fourfold  $\tilde{X}_4$  inside  $X_6$  is given as  $(4F + H)(3F + H) = 6F(2F + H)$ . On  $\tilde{X}_4$ , the first relation becomes

$$F(F + K) = 0. \quad (4.39)$$

The standard section of the Weierstrass model at  $z = 0$  is still present and sits at  $y = X = \lambda_1 = \lambda_2 = 1$ . Furthermore, there is now a second section at  $\lambda_1 = X = Y_- = 0$ . Such models should be related to elliptic fibrations of  $E_7$  type, which generically have a second section [72–77]. In [67], the appearance of a fibration of  $E_7$  type was shown by blowing up the singular fourfold. It would be interesting to establish a similar result also for the small resolution considered here.

#### 4.2.2 The flux after the transition

We now want to see how to construct a four-form flux with one leg in the fiber in the case of  $\tilde{X}_4$ . We can consider cycles given by

$$\sigma_{\alpha} = \{\lambda_1 = 0\} \cap \{Y_- = 0\} \cap \{X = 0\} \cap \{\alpha = 0\} \subset X_6. \quad (4.40)$$

This is the intersection of the new section of the elliptic fibration with the locus  $\alpha = 0$ .

First, we demand that the corresponding flux have a vanishing intersection with any two divisors of the base. As before we have

$$\int_{\tilde{X}_4} D_i \cdot D_j \cdot \sigma_{\alpha} = \int_{B_3} D_i \cdot D_j \cdot [\alpha]. \quad (4.41)$$

Furthermore, we want the flux to be orthogonal to the  $z = 0$  section. We still have

$$\sigma_{\alpha} \cdot (F + K) = 0, \quad (4.42)$$

as  $x = y = z = 0$  is part of the exceptional set.

As before, we can cancel the term arising in (4.41) by subtracting  $[\alpha]$  times  $F$ :

$$\gamma_\alpha = \sigma_\alpha - F \cdot [\alpha], \quad (4.43)$$

because

$$\int_{\tilde{X}_4} F \cdot [\alpha] \cdot D_i \cdot D_j = (F + K) \cdot [\alpha] \cdot D_i \cdot D_j = \int_{B_3} [\alpha] \cdot D_i \cdot D_j, \quad (4.44)$$

where we have used the section of the elliptic fibration to restrict the integral to the base.

Furthermore,

$$F \cdot [\alpha] \cdot (F + K) = 0 \quad (4.45)$$

holds inside  $\tilde{X}_4$ , so that  $\gamma_\alpha$  fulfills both requirements (4.41), (4.42). Note that even though  $\gamma_\rho$  was trivial in the ambient space  $X_5$ ,  $\gamma_\alpha$  is non-trivial in the ambient space  $X_6$ .

Let us now compute the tadpole of a flux  $G_\alpha = \gamma_\alpha$  on  $\tilde{X}_4$ . First note that

$$\sigma_\alpha \cdot (F + K) = 0. \quad (4.46)$$

This equation is obtained as before:  $F + K$  forces  $z = 0$ , whereas  $\sigma_\alpha$  is along  $z = 1$ . As we also have  $G_\alpha \cdot D_i \cdot D_j = 0$ , it again follows that  $G_\alpha \cdot F = 0$ . Proceeding as before, we find

$$\begin{aligned} Q_\alpha &= -\frac{1}{2} \int_{\tilde{X}_4} G_\alpha \wedge G_\alpha = -\frac{1}{2} \int_{\tilde{X}_4} G_\alpha \wedge \sigma_\alpha \\ &= -\frac{1}{2} \int_{\tilde{X}_4} (\sigma_\alpha \wedge \sigma_\alpha - \sigma_\alpha \cdot F \cdot [\alpha]) \\ &= -\frac{1}{2} \int_{\tilde{X}_4} \sigma_\alpha \wedge \sigma_\alpha + \frac{1}{2} \int_{B_3} \bar{K} \cdot [\alpha]^2. \end{aligned} \quad (4.47)$$

To compute the self-intersection of  $\sigma_\alpha$  inside  $\tilde{X}_4$ , we note (as before) that

$$\int_{\tilde{X}_4} \sigma_\alpha \wedge \sigma_\alpha = \int_{\sigma_\alpha} c_2(N_{\sigma_\alpha \subset \tilde{X}_4}). \quad (4.48)$$

We can compute the second Chern class of the normal bundle of  $\sigma_\alpha$  by using  $N_{\tilde{X}_4 \subset X_6} = (4F + H) \oplus (3F + H)$ ,  $N_{\sigma_\alpha \subset X_6} = 2F \oplus 3F \oplus [\alpha] \oplus H$  and

$$c(N_{\sigma_\alpha \subset \tilde{X}_4}) = c(N_{\sigma_\alpha \subset X_6}) / c(N_{\tilde{X}_4 \subset X_6}) \quad (4.49)$$

which yields

$$c(N_{\sigma_\alpha \subset \tilde{X}_4})_2 = 8F^2 + 9F \cdot H + H^2 - 2F \cdot \alpha - H \cdot \alpha \quad (4.50)$$

Hence

$$\begin{aligned} \int_{\sigma_\alpha} c_2(N_{\sigma_\alpha \subset \tilde{X}_4}) &= \int_{X_6} 6F^2 \cdot H \cdot [\alpha] \cdot c_2(N_{\sigma_\alpha \subset \tilde{X}_4}) \\ &= - \int_{X_6} 6F^2 \cdot H \cdot [\alpha] \cdot \bar{K} \cdot [\alpha] \\ &= - \int_{B_3} [\alpha]^2 \bar{K}, \end{aligned} \quad (4.51)$$

where we have repeatedly used Poincaré duality and the relations coming from the SR-ideal.

Putting everything together, and using  $\bar{K} = c_1(B_3)$ ,

$$Q_\alpha = -\frac{1}{2} \int_{\tilde{X}_4} \sigma_\alpha \wedge \sigma_\alpha + \frac{1}{2} \int_{B_3} \bar{K} \cdot [\alpha]^2 = \int_{B_3} c_1(B_3) \cdot [\alpha]^2. \quad (4.52)$$

Looking back at (4.33), we conclude that after the transition to intersection branes, (4.28), a flux  $G_\rho$  on  $X_4$  is replaced by a flux  $G_\alpha$  on  $\tilde{X}_4$  as in (4.43), with  $[\alpha] = [\rho] - 3\bar{K}$ . We will confirm this result from the perspective of type IIB string theory in section 5.2.

### 4.3 Chirality

As two-form fluxes on branes can induce chirality, and the flux  $G_4$  that we have constructed is related to such fluxes, we expect it to also induce chirality. In F-theory, matter resides at the intersections of branes, the so-called matter curves. Over these curves, the ADE singularity sitting over a 7-brane is enhanced. This enhancement is related to extra vanishing  $\mathbb{P}^1$ s over the matter curve. They extend the Dynkin diagram of the ADE gauge group on the 7-brane, which allows to determine the representation of matter that sits on the matter curve.

If we fiber the vanishing  $\mathbb{P}^1$ s responsible for charged matter over their matter curves  $C_i$ , we obtain four-cycles  $\hat{C}_i$  of  $X_4$ . It hence seems natural that the chirality can be obtained as the integral of  $G_4$  over these curves [3, 6]. In this section, we will test this idea by explicitly computing the integral of our proposed fluxes,

$$I = \int_{\hat{C}_i} G_4, \quad (4.53)$$

after appropriately resolving the matter curves.

#### 4.3.1 A simple example

The simplest situation in which a matter curve arises is the configuration considered in section 4.2. In the base, the matter curve is simply given by  $a_3 = b_4 = 0$ . The corresponding four-cycle  $\hat{C}$  is the fibration of the  $\mathbb{P}^1$  introduced in (4.36) over this curve. It is given by  $a_3 = b_4 = X = y = 0$  in the ambient sixfold  $X_6$  introduced in section 4.2.1.

Let us hence consider a flux  $G_4 = \gamma_\alpha$  as before and compute the intersection number

$$\int_{\hat{C}} \gamma_\alpha = \int_{\hat{C}} (\sigma_\alpha - F \cdot [\alpha]). \quad (4.54)$$

We first consider the second term. Since  $xyz$  is in the SR-ideal of  $X_5$ , the intersection of  $\hat{C}$  with  $z = 0$  is empty. Hence,  $\hat{C} \cdot (F + K) = 0$ , so that the second term is equal to  $\int_{X_4} \hat{C} \cdot [\alpha] \cdot K$ . Since this involves intersecting four polynomials that depend on  $B_3$  only, this intersection must be empty. As both the cycle  $\sigma_\alpha$ , (4.40), and the curve  $\hat{C}$ , lie within the section over  $Y_- = X = 0$ , we find

$$\int_{\hat{C}} G_\alpha = \sigma_\alpha \cdot \hat{C} = 12 \int_{B_3} c_1(B_3)^2 \cdot [\alpha]. \quad (4.55)$$

In the presence of such  $G_4$  flux, we claim that this is the chirality index that counts the massless M2-brane states wrapped on the  $\mathbb{P}^1$  hovering over  $C$ . We will validate this result in the weak coupling limit by performing the corresponding computation in the IIB picture in section 5.2.

## 4.4 Non-abelian singularities, fluxes and chirality

### 4.4.1 $Sp(1)$ singularity

As a further example, we consider the Weierstrass equation in the case of an  $Sp(1)$  singularity along the locus  $P = 0$  in the base<sup>5</sup>. Hence, the discriminant factorizes into an  $Sp(1)$ -brane locus, and a remaining  $I_1$ -locus. This means that the polynomials  $a_3, a_4, a_6$ , and correspondingly  $b_4$  and  $b_6$ , factorize in the following way:

$$a_3 = a_{3,1} \cdot P \quad a_4 = a_{4,1} \cdot P \quad a_6 = a_{6,2} \cdot P^2 \quad b_4 = b_{4,1} \cdot P \quad b_6 = b_{6,2} \cdot P^2. \quad (4.56)$$

We will put the further constraint  $a_{6,2} \equiv \rho\tau$  on the complex structure in order to define a four-form flux as in the previous sections.

To resolve the singularity over  $P = 0$ , we follow [83] and introduce a new coordinate  $s$  and a new equation  $s = P$ . Then the singular fourfold  $X_4$  is given by the two equations:

$$\begin{cases} Y_+ Y_- - X Q - s^2 \rho\tau z^6 = 0 \\ s = P \end{cases} \quad (4.57)$$

with

$$\begin{aligned} Y_{\pm} &= y \pm \frac{1}{2} z^3 a_{3,1} s \\ X &= x - \frac{1}{12} z^2 b_2 \\ Q &= X(X + \frac{1}{4} z^2 b_2) + \frac{1}{2} z^4 b_{4,1} s. \end{aligned} \quad (4.58)$$

The resolved fourfold  $\hat{X}_4$  is then given by the two equation

$$\begin{cases} Y_+ Y_- - X Q_v - s^2 \rho\tau z^6 = 0 \\ s v = P \end{cases} \quad (4.59)$$

in an ambient sixfold  $X_6$ . Note that homogeneity of the equations enforces that the definition of  $Q$  is now changed to  $Q_v = X(v X + \frac{1}{4} z^2 b_2) + \frac{1}{2} b_{4,1} s z^4$ . The new exceptional divisor is  $E : v = 0$ . The coordinates  $x, y, z, s, v$  belong to  $F^2 \otimes [E]^{-1}$ ,  $F^3 \otimes [E]^{-1}$ ,  $F \otimes K$ ,  $[P] \otimes [E]^{-1}$ ,  $[E]$ , which is summarized in the table below.

$x$	$y$	$z$	$s$	$v$
0	0	$K$	$[P]$	0
2	3	1	0	0
-1	-1	0	-1	1

The SR-ideal of the resolved ambient six-fold is

$$\{xyz, Xys, vz\}. \quad (4.60)$$

---

<sup>5</sup> Though  $Sp(1) \cong SU(2)$ , we will distinguish between an  $Sp(1)$  and an  $SU(2)$  singularity, meaning with the first a non-split  $SU(2)$  and with the second a split one.

From the SR-ideal we see that the following relations hold:

$$F^2(F + \bar{K}) = 0, \quad (E - [P])(3F - E)(2F - E), \quad E(F + K) = 0. \quad (4.61)$$

In particular, we note that on  $B_3$  we have  $F = \bar{K}$  and  $E = [P]$ .

Let us now see what are the possible holomorphic four-cycles that we can have on the resolved fourfold. First, we have a cycle like in the smooth case:

$$\gamma_\rho = \sigma_\rho - [\rho] \left( F - \frac{E}{2} \right) [\hat{X}_4] \quad (4.62)$$

where

$$\sigma_\rho = \{ \rho = 0, Y_- = 0, X = 0, vs = P \} |_{X_6} \subset \hat{X}_4. \quad (4.63)$$

We have chosen the subtraction such that the flux is trivial in the ambient space, while it is non-trivial on the fourfold. This choice is motivated by the observation that this naturally happens for the corresponding configuration without an  $Sp(1)$  stack. One can check that  $\gamma_\rho$  is zero when  $\rho$  has maximal degree or is a constant.

Let us compute the tadpole of a flux  $G_4^{(I_1)} = \gamma_\rho$ :

$$-\frac{1}{2} \int_{\hat{X}_4} G_4^{(I_1)} \wedge G_4^{(I_1)} = -\frac{1}{2} \int_{\hat{X}_4} \sigma_\rho \cdot \sigma_\rho - \frac{1}{2} \int_{\hat{X}_4} \sigma_\rho \cdot [\rho] \left( F - \frac{E}{2} \right) [\hat{X}_4]. \quad (4.64)$$

We compute the second piece first:

$$-\frac{1}{2} \int_{\hat{X}_4} \sigma_\rho \cdot [\rho] \left( F - \frac{E}{2} \right) [\hat{X}_4] = -\frac{1}{2} \int_{B_3} [\rho]^2 \left( \bar{K} - \frac{[P]}{2} \right). \quad (4.65)$$

The first piece in (4.64) is computed with the trick of normal bundles, i.e.

$$-\frac{1}{2} \int_{\hat{X}_4} \sigma_\rho \cdot \sigma_\rho = -\frac{1}{2} \int_{\sigma_\rho} c_2(N_{\sigma_\rho \subset \hat{X}_4}), \quad (4.66)$$

where the normal bundle is determined by the following exact sequence

$$0 \rightarrow N_{\sigma_\rho \subset \hat{X}_4} \rightarrow N_{\sigma_\rho \subset X_6} \rightarrow N_{\hat{X}_4 \subset X_6} \rightarrow 0. \quad (4.67)$$

We find that:

$$\begin{aligned} c(N_{\sigma_\rho \subset \hat{X}_4}) &= \frac{c(N_{\sigma_\rho \subset X_6})}{c(N_{\hat{X}_4 \subset X_6})} \\ &= 1 + \{[\rho] - F\} + \{12F^2 - 7EF + E^2 - [\rho]F\}, \end{aligned} \quad (4.68)$$

consequently

$$-\frac{1}{2} \int_{\hat{X}_4} \sigma_\rho \cdot \sigma_\rho = -\frac{1}{2} \int_{B_3} [\rho] (12\bar{K}^2 - 7[P]\bar{K} + [P]^2 - [\rho]\bar{K}). \quad (4.69)$$

Putting everything together:

$$\begin{aligned} -\frac{1}{2} \int_{\hat{X}_4} G_4^{(I_1)} \wedge G_4^{(I_1)} &= \int_{B_3} ([\rho]\bar{K} - \frac{1}{4}[\rho][P] - \frac{1}{2}[P]^2 + \frac{7}{2}[P]\bar{K} - 6\bar{K}^2) \\ &= -\int_{B_3} (c_1(B_3) - \frac{1}{4}[P]) \cdot [\rho] \cdot (6c_1(B_3) - 2[P] - [\rho]), \end{aligned} \quad (4.70)$$

where we have substituted  $\bar{K} = c_1(B_3)$ . Note that for  $[P] = 0$  we get the result of the previous section.

As we have resolved the  $Sp(1)$  singularity over  $P = 0$ , we should be able to construct a four-form flux which corresponds to a flux in the Cartan. This flux has the form:

$$G_4^{(\text{Sp})} = \frac{1}{2}E \wedge [q] \quad (4.71)$$

where  $[q]$  is the divisor  $q = 0$  for some polynomial  $q$  of the base. This flux is integral if  $[q]$  is even and half-integral otherwise. The cancellation of the Witten anomaly,

$$G_4 + \frac{c_2(\hat{X}_4)}{2} \in H^4(\hat{X}_4, \mathbb{Z}), \quad (4.72)$$

can enforce the introduction of this flux, i.e. it determines if  $q$  is odd or even. In [83], it was shown that any smooth elliptically fibered CY fourfold with a Weierstrass representation has an even  $c_2$ . It was also argued there that this implies that any odd contribution to  $c_2$  must come from holomorphic submanifolds that do not survive the blow down map of the resolution of a singularity.

The flux  $G_4^{(\text{Sp})}$  is Poincaré invariant: Its orthogonality to the section  $z = 0$  is implied by the SR-ideal, i.e.  $E(F + K) = 0$ . The intersection of  $G_4^{(\text{Sp})}$  with  $D_i \cdot D_j$  is zero for the following reason: The divisor  $E$  is a  $\mathbb{P}^1$  fibration over a four-cycle on the base; the wedge product  $[q] \wedge D_i \wedge D_j$  is Poincaré dual to the fiber on some point on the base that in general will not belong to the base of  $E$ . Hence the generic intersection is empty.

Let us compute the  $D3$ -charge of this flux:

$$-\frac{1}{2} \int_{\hat{X}_4} G_4^{(\text{Sp})} \wedge G_4^{(\text{Sp})} = -\frac{1}{2} \int_{\hat{X}_4} \frac{1}{4} E^2 [q]^2 = \int_{B_3} [P] \left( \frac{[q]}{2} \right)^2. \quad (4.73)$$

The flux  $G_4^{(\text{Sp})}$  is orthogonal to  $G_4^{(I_1)}$ , i.e.  $\int_{\hat{X}_4} G_4^{(\text{Sp})} \wedge G_4^{(I_1)} = 0$ . Hence the total tadpole is the sum of the two.

The resolution we made creates a new four-cycle which is a fibration over the curve where the  $Sp(1)$ -brane intersects the remaining component of the discriminant locus. This four-cycle is given by two equations in  $\hat{X}_4$  (see appendix C for details):

$$\hat{C} : \quad \{v = 0\} \cap \{b_{4,1}^2 - b_2(a_{3,1}^2 + 4\rho\tau) = 0\}. \quad (4.74)$$

The second equation is a section of  $\bar{K}^8 [P]^{-2}$ .

We claim that the number of 4D chiral states coming from the intersection of the  $Sp(1)$  surface with the remaining brane is the integral of the four-form flux over the four-cycle  $\hat{C}$ . Indeed, this four-cycle arises, after resolution, at the locus where the singularity enhances.

If we integrate the flux  $G_4^{(I_1)}$  on  $\hat{C}$ , we will get zero, as  $\hat{C}$  coincides with its pushforward to the ambient space  $X_6$ . Hence the integral of  $G_4 = G_4^{(I_1)} + G_4^{(\text{Sp})}$  on  $\hat{C}$  is equal to the integral of  $G_4^{(\text{Sp})}$ :

$$\begin{aligned} \int_{\hat{C}} G_4 &= \int_{\hat{C}} G_4^{(\text{Sp})} = \frac{1}{2} \int_{\hat{X}_4} E^2[q] [b_{4,1}^2 + b_2(4\rho\tau - a_{3,1}^2)] \\ &= - \int_{B_3} [P] \cdot [q] \cdot (8c_1(B_3) - 2[P]), \end{aligned} \quad (4.75)$$

where we have used that  $[\hat{X}_4] \cdot E^2 = -2[P] \cdot [B_3]$ .

Let us finally propose a formula for computing the bulk chiral matter, which is the matter living on the  $Sp(1)$ -brane at  $P = 0$ :

$$\int_E G_4 \wedge (c_1(E) - 2\bar{K}) \quad (4.76)$$

in analogy with the type IIB formula

$$\frac{1}{2} (\langle \Gamma', \Gamma \rangle + \frac{1}{4} \langle \Gamma_{\text{O}7}, \Gamma \rangle) = - \int_{[D7]} F \cdot (c_1([D7]) + \bar{K}) .$$

$E$  is the exceptional divisor that is a  $\mathbb{P}^1$  fibration over the surface wrapped by the  $Sp(1)$  brane. In our case:

$$\int_E G_4 \wedge (c_1(E) - 2\bar{K}) = - \int_{\hat{X}_4} G_4^{(\text{Sp})} (E^2 + 2\bar{K}) = \int_{B_3} [q][P]([P] - \bar{K}) . \quad (4.77)$$

In the weak coupling limit, the results of this section are confirmed from the type IIB perspective in section 5.3.

#### 4.4.2 $Sp(1)$ in a $U(1)$ restricted model

Let us consider the same situation as in the last section and make a deformation such that  $\rho \equiv 0 \equiv \tau$ . This fourfold has an  $Sp(1)$  singularity along the surface  $S$ :  $X = y = P = 0$  and an additional  $SU(2)$  singularity along the curve  $C$ :  $X = y = a_{3,1} = b_{4,1} = 0$ .<sup>6</sup>

Let us resolve the fourfold along the  $Sp(1)$  singularity like in Section 4.4.1. It is then described by the equations:

$$\begin{cases} Y_+ Y_- - X Q_v = 0 \\ s v = P \end{cases} \quad (4.78)$$

with  $X, Y_{\pm}$  like in (4.58) and  $Q_v = X(vX + \frac{1}{4}b_2z^2) + \frac{1}{2}b_{4,1}z^4s$ . The new exceptional divisor is  $E : v = 0$ . The coordinates  $x, y, z, s, v$  belong to  $F^2 \otimes [E]^{-1}$ ,  $F^3 \otimes [E]^{-1}$ ,  $F \otimes K$ ,  $[P] \otimes [E]^{-1}$ ,  $[E]$ .

---

<sup>6</sup>If we try to make a small resolution to cure the singularity on the curve  $C$  (in the same way we did in section 4.2.1, we also resolve the singularity along the surface  $S$ . One can look for remaining singularities on this resolved fourfold. Indeed, there is a singularity along the curve  $X = y = P = b_{4,1}\lambda_1 - a_{3,1}\lambda_2 = \lambda_2^2 - b_2\lambda_1^2 = 0$ .



One can see that the resolved fourfold still has a singularity along the curve  $X = y = a_{3,1} = b_{4,1} = 0$ , which we cure with a small resolution. The resulting fourfold  $\tilde{X}_4$  is given by

$$\tilde{X}_4 : \begin{cases} Q_v \lambda_1 = Y_- \lambda_2 \\ X \lambda_2 = Y_+ \lambda_1 \\ v s = P \end{cases} \subset X_7 \quad (4.79)$$

where  $\lambda_1, \lambda_2$  are coordinates on a  $\mathbb{P}^1$ . Their divisor classes are  $[\lambda_1] = H$  and  $[\lambda_2] = H + F$ . These satisfy  $H(H + K) = 0$ .

We have summarized the divisor classes of the toric coordinates in the table below.

$x$	$y$	$z$	$s$	$v$	$\lambda_1$	$\lambda_2$
0	0	$K$	$[P]$	0	0	0
2	3	1	0	0	0	1
-1	-1	0	-1	1	0	0
0	0	0	0	0	1	1

Let us see what are the possible holomorphic four-form fluxes that we can have on this resolved fourfold. First, we can have an algebraic four-cycle as before

$$\gamma_\alpha = \sigma_\alpha - [\alpha](F - \tfrac{1}{2}E)[\tilde{X}_4] \quad (4.80)$$

where

$$\sigma_\alpha = \{\alpha = 0, Y_- = 0, X = 0, \lambda_1 = 0, v s = P\}. \quad (4.81)$$

The four-form subtraction is chosen such that the corresponding flux  $G_4^{(I_1)}$  is Poincaré invariant and orthogonal to all fluxes of the type  $G_4^{(\text{Sp})}$  (see below).

The form  $\gamma_\alpha$  is not trivial in  $X_7$ , although it is Poincaré invariant. Its homology class in  $X_7$  is:

$$\gamma_\alpha = [\alpha] (H - (F - \tfrac{1}{2}E)) [\tilde{X}_4]. \quad (4.82)$$

Moreover, when we compute the tadpole of this flux (using again the normal bundle trick), we find that

$$-\tfrac{1}{2} \int G_4^{(I_1)} \wedge G_4^{(I_1)} = \int_{B_3} [\alpha]^2 \cdot \left( \bar{K} - \frac{[P]}{4} \right) = \int_{B_3} [\alpha]^2 \cdot \left( c_1(B_3) - \frac{[P]}{4} \right). \quad (4.83)$$

Again, we can have a flux in the Cartan of  $Sp(1)$ :

$$G_4^{(\text{Sp})} = \tfrac{1}{2}E \wedge [q]. \quad (4.84)$$

This flux induces the same tadpole as before, (4.73).

In this setup we have several intersections between brane-loci: There is the intersection between the  $Sp(1)$  locus and the remaining 7-brane, as well as the intersection of the remaining 7-brane with itself. This last curve produces a four-cycle as before:

$$\hat{C}_1 : \{X = 0, y = 0, b_{4,1} = 0, a_{3,1} = 0, v s = P\} \text{ in } X_7. \quad (4.85)$$

On the other hand, the four-cycle coming from the resolution of the singularity over the curve  $C_2 : \{P = b_{4,1}^2 - b_2 a_{3,1}^2 = 0\}$  is the sum of two four-cycles, as explained in appendix C. This sum is given by

$$\hat{C}_2 : \{v = 0, b_{4,1}^2 - b_2 a_{3,1}^2 = 0\} \text{ in } \tilde{X}_4. \quad (4.86)$$

It appears as two  $\mathbb{P}^1$ s on top of the curve  $C_2$ . Thinking in terms of the weak coupling limit, we can gain some intuition about the meaning of the different  $\mathbb{P}^1$ s of the resolution. Let us denote the two 7-branes of the  $Sp(1)$  stack by  $S$  and  $S'$  and the remaining ‘ $U(1)$ -restricted’ 7-branes by  $D_{I_1}$ . One  $\mathbb{P}^1$  gives M2-branes associated with strings  $D_{I_1} \rightarrow S$ , while the other  $\mathbb{P}^1$  is the one wrapped by M2-branes associated with strings  $S' \rightarrow D_{I_1}$ . There exist two different four-cycles that are the fibration of only one of these  $\mathbb{P}^1$ s over the curve  $C_2$ . The four-cycle  $\hat{C}_2$  is the sum of them.

Even though we can describe the sum of those two  $\mathbb{P}^1$ s as an algebraic cycle, we cannot see them independently in an algebraic way. We can achieve this, however, by constructing an auxiliary fourfold: We introduce a new coordinate  $\xi$  living in  $\tilde{K}$ , and a new equation  $\xi^2 = b_2$ <sup>7</sup>. The auxiliary fourfold (which is no longer a CY), is a double cover of the CY fourfold given by the equations:

$$\tilde{Y}_4 : \begin{cases} \begin{pmatrix} X(vX + \frac{1}{4}\xi^2 z^2) + \frac{1}{2}b_{4,1}s z^4 & -Y_m \\ -Y_p & X \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ vs = P \\ \xi^2 = b_2 \end{cases} \quad (4.87)$$

embedded in an ambient eightfold  $X_8$ . In  $X_8$ , the equation defining the curve  $C_2$  factorizes:

$$C_2 \rightarrow C_{2,+} \cup C_{2,-} \quad \text{where} \quad C_{2,\pm} : \{P = 0, b_{4,1} = \pm \xi a_{3,1}\}. \quad (4.88)$$

In this double cover  $\tilde{Y}_4$ , the ‘ $U(1)$ -restricted’  $I_1$  brane on  $D_{I_1}$  lifts to two branches:

$$D \cup D' \mapsto D_{I_1}. \quad (4.89)$$

From the IIB perspective, these can be thought of as the brane and its orientifold image. The situation is summarized in figure 1.

Consider the locus  $\{v = 0, b_{4,1} = \xi a_{3,1}\}$ : The determinant of the  $2 \times 2$  matrix in (4.87) factorizes:

$$\det \begin{pmatrix} \frac{1}{4}\xi^2 z^2 X + \xi b_{4,1}s z^4 & -y + \frac{1}{2}a_{3,1}s z^3 \\ -y - \frac{1}{2}a_{3,1}s z^3 & X \end{pmatrix} = (y - \Upsilon_{2,+})(y + \Upsilon_{2,+}) \quad (4.90)$$

where  $\Upsilon_{2,+} = \frac{1}{2}a_{3,1}s z^3 + \frac{1}{2}\xi z X$ . On each of the two branches, the two lines of the matrix become dependent and only give one more relation. Hence we get the following two four-cycles in  $X_8$  (that belong to  $\tilde{Y}_4$ ):

$$\hat{C}_{2,+}^{(D \rightarrow S)} = \{v = 0, b_{4,1} = \xi a_{3,1}, y = \Upsilon_{2,+}, X\lambda_2 = (\frac{1}{2}\xi z X + 2z^3 \psi s)\lambda_1, vs = P, \xi^2 = b_2\} \quad (4.91)$$

$$\hat{C}_{2,+}^{(S' \rightarrow D)} = \{v = 0, b_{4,1} = \xi a_{3,1}, y = -\Upsilon_{2,+}, \lambda_2 = -\frac{1}{2}\xi z \lambda_1, vs = P, \xi^2 = b_2\}.$$

<sup>7</sup>From the weak coupling limit, it is clear what this means: We construct the double cover of the base space, branched over the location of the  $O7$ -plane. As we have chosen  $b_{6,2}$  to be a square, this means that  $\Delta_{12,2}$  describes a brane and its image (under the corresponding involution) which are *separate* surfaces in the double cover.

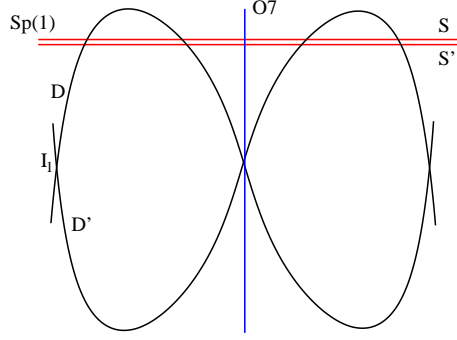


Figure 1: Setup with an ‘ $U(1)$ -restricted’  $I_1$  brane, and an  $Sp(1)$ -stack.

It is crucial that these two four-cycles are in different homology classes.

Let us integrate  $\sigma_\alpha$  over the two four-cycles:

$$\begin{aligned} \int_{\hat{C}_{2,+}^{(D \rightarrow S)}} \sigma_\alpha &= \int_{X_8} [\alpha][y][X][\lambda_1] E[b_{4,1} - \xi a_{3,1}][vs - P][\xi^2 - b_2] \\ &= \int_{X_3} [\alpha][b_{4,1} - \xi a_{3,1}][P] = \int_{X_3} [\alpha](4\bar{K} - [P])[P]. \end{aligned} \quad (4.92)$$

Where we have used that  $X = 0$ ,  $Y_- = 0$  and  $\lambda_1 = 0$  in  $\sigma_\alpha$  satisfy  $X\lambda_2 = (\frac{1}{2}\xi X + 2a_{3,1}s)\lambda_1$  and  $y = a_{3,1}s + \frac{1}{2}\xi X$  in  $\hat{C}_1^{(D \rightarrow S)}$ .

The integral on the other four-cycle vanishes:

$$\int_{\hat{C}_{2,+}^{(S' \rightarrow D)}} \sigma_\alpha = 0, \quad (4.93)$$

as  $\lambda_1 = 0$  in  $\sigma_\alpha$  does not intersect  $\lambda_2 = -\frac{1}{2}\xi\lambda_1$  in  $\hat{C}_1^{(S' \rightarrow D)}$ .

Let us now compute the integral of the four-form  $[p] \wedge E$  over the two four-cycles under study. Here  $p$  is any polynomial in  $B_3$ . We claim that we get the same result from the integral of this form on the two four-cycles. In fact the homology classes of the two four-cycles in  $X_8$  differ by the class:

$$\hat{C}_{2,+}^{(D \rightarrow S)} - \hat{C}_{2,+}^{(S' \rightarrow D)} = [X][y]E[b_{4,1} - \xi a_{3,1}][vs - P][\xi^2 - b_2]. \quad (4.94)$$

If we integrate  $[p] \wedge E$  over it, we get

$$\int_{\hat{C}_{2,+}^{(D \rightarrow S)} - \hat{C}_{2,+}^{(S' \rightarrow D)}} [p] \wedge E = \int_{X_8} [X][y]E[P]^2[p][b_{4,1} - \xi a_{3,1}][\xi^2 - b_2] = 0 \quad (4.95)$$

because we have the intersection of four divisors of  $B_3$ <sup>8</sup>.

<sup>8</sup>This is a bit subtle. In general the class  $[P]$  can also contain the variables  $v, s$ ; but when we intersect with  $E[X][y]$ , we are fixing the variables  $v, s$  so that this class only describes a divisor in the base  $B_3$ .

So we are left with the integral

$$\begin{aligned}
\int_{\hat{C}_{2,+}^{(S' \rightarrow D)}} [p] \wedge E &= \int_{X_8} [p][b_{4,1} - \xi a_{3,1}][y]E[vs - P][\lambda_2]E[\xi^2 - b_2] \\
&= \int_{X_8} [p][b_{4,1} - \xi a_{3,1}][y]E[vs - P][\lambda_2](-[X])(\xi^2 - b_2) \\
&= - \int_{X_3} [p](4\bar{K} - [P])[P]
\end{aligned} \tag{4.96}$$

where we used  $[X] = -E + 2F$  and the fact that  $EF = E\bar{K}$ .

With the help of these integrals and remembering that  $\bar{K} = c_1(B_3)$ , we can compute the integral of  $G_4^{(I_1)}$  on the two four-cycles:

$$\begin{aligned}
\int_{\hat{C}_{2,+}^{(D \rightarrow S)}} G_4^{(I_1)} &= \int_{\hat{C}_{2,+}^{(D \rightarrow S)}} \sigma_\alpha - \int_{\hat{C}_{2,+}^{(D \rightarrow S)}} s \\
&= \int_{\hat{C}_{2,+}^{(D \rightarrow S)}} \sigma_\alpha + \frac{1}{2} \int_{\hat{C}_{2,+}^{(D \rightarrow S)}} [\alpha]E \\
&= \frac{1}{2} \int_{X_3} [\alpha](4c_1(B_3) - [P])[P],
\end{aligned} \tag{4.97}$$

$$\begin{aligned}
\int_{\hat{C}_{2,+}^{(S' \rightarrow D)}} G_4^{(I_1)} &= \int_{\hat{C}_{2,+}^{(S' \rightarrow D)}} \sigma_\alpha - \int_{\hat{C}_{2,+}^{(S' \rightarrow D)}} s \\
&= \frac{1}{2} \int_{\hat{C}_{2,+}^{(S' \rightarrow D)}} [\alpha]E \\
&= -\frac{1}{2} \int_{X_3} [\alpha](4c_1(B_3) - [P])[P].
\end{aligned} \tag{4.98}$$

In this case we see that the number of M2-branes corresponding to strings  $D \rightarrow S$  is the same as those corresponding to strings  $D \rightarrow S'$ . These states form a **2** of  $Sp(1)$ . This matches with the fact that we have switched on a two-form flux only on the  $U(1)$  7-brane. This does not break the gauge group on the  $Sp(1)$  stack. Nonetheless, it gives rise to chiral states that are charged under the  $U(1)$  and furthermore sit in a representation of the gauge group  $Sp(1)$ .

If we also allow for a flux  $G_4^{(\text{Sp})}$  as in (4.84), the chiral indices receive the contribution

$$\int_{\hat{C}_{2,+}^{(D \rightarrow S)}} G_4^{(\text{Sp})} = \int_{\hat{C}_{2,+}^{(S' \rightarrow D)}} G_4^{(\text{Sp})} = \frac{1}{2} \int_{\hat{C}_{2,+}^{(S' \rightarrow D)}} [q]E = -\frac{1}{2} \int_{X_3} [q](4\bar{K} - [P])[P]. \tag{4.99}$$

Putting everything together, one realizes that a flux  $G_4 = G_4^{(I_1)} + G_4^{(\text{Sp})}$  induces the chiralities

$$\begin{aligned}
\int_{\hat{C}_{2,+}^{(D \rightarrow S)}} G_4 &= \int_{X_3} \left( \frac{[\alpha]}{2} - \frac{[q]}{2} \right) (4c_1(B_3) - [P])[P], \\
\int_{\hat{C}_{2,+}^{(S' \rightarrow D)}} G_4 &= - \int_{X_3} \left( \frac{[\alpha]}{2} + \frac{[q]}{2} \right) (4c_1(B_3) - [P])[P].
\end{aligned} \tag{4.100}$$

We will see a perfect match with the perturbative type IIB formulae in section 5.4.

Finally, we note that there are two other four-cycles which come from the curve at  $v = 0$  and  $b_4 = -\xi a_3$ . These two four-cycles are in the same homology classes as the ones studied and hence give the same result. They are mapped to the cycles already considered under the involution  $\xi \rightarrow -\xi$ .

We can also consider the singlets under  $Sp(1)$  which are located at the intersection of the  $U(1)$  brane with itself. The computation of section 4.3.1 carries over to the present case, so that we find that the flux  $G_4^{(I_1)}$  gives a chiral spectrum:

$$\int_{\hat{C}_1} G_4^{(I_1)} = \int_{\hat{C}_1} \sigma_\alpha = \int_{B_3} (4c_1(B_3) - [P]) \cdot (3c_1(B_3) - [P]) \cdot [\alpha]. \quad (4.101)$$

In a GUT model, we hence expect that our flux will also *generate chirality for GUT singlets*.

#### 4.4.3 $SU(N)$ GUT models and chiral matter

In this section we want to give an example that is closer to the  $SU(5)$  GUT constructions in F-theory. In those models, one has a fourfold with an  $SU(5)$  singularity along a divisor  $S_{GUT}$  at  $\zeta = 0$ . Along curves in  $S_{GUT}$  the singularity can enhance to  $SU(6)$  or  $SO(10)$ . On these curves new degrees of freedom arise. These loci can be seen to be the intersection between  $S_{GUT}$  with the remaining 7-brane  $I_1$  locus. In particular, the discriminant of the Weierstrass equation is required to factorize

$$\Delta \sim \zeta^5 \cdot p \quad (4.102)$$

where  $p$  is a polynomial such that  $[p] + 5[\zeta] = 12\bar{K}$ . The locus at which  $\{\zeta = 0\} \cap \{p = 0\}$  factorizes in two pieces, one with an  $SU(6)$  enhancement, over which we have matter in the **5** of  $SU(5)$ , an one with  $SO(10)$  enhancement, where we have matter in the **10**. If we want to preserve the GUT group, then no flux in the Cartan of the  $U(5)$  stack should be switched on. On the other hand, if we want 4D chiral matter in the  $\bar{\mathbf{5}} \oplus \mathbf{10}$ , then a flux must be switched on either along the diagonal  $U(1)$  of the  $U(5)$  (as in [66]) or along the  $U(1)$  of the  $U(1)$ -restricted  $I_1$ -locus. We proceed with the second approach and we switch on a flux of kind  $G_4^{(I_1)}$ .

We now describe a simplified example: We consider a split  $SU(2)$  singularity along a divisor  $P = 0$ , instead of a split  $SU(5)$ . This case is realized when the sections  $a_i$  factorize in the following way:

$$a_2 = P \cdot a_{2,1} \quad a_3 = P \cdot a_{3,1} \quad a_4 = P \cdot a_{4,1} \quad a_6 = P^2 \cdot a_{6,2}. \quad (4.103)$$

With respect to the  $Sp(1)$  case we now also have factorized  $a_2$ . This leads to a non-generic form of  $b_2$ :

$$b_2 = a_1^2 + 4a_{2,1} \cdot P. \quad (4.104)$$

The discriminant of the Weierstrass equation is factorized as  $\Delta = P^2(b_{4,1}^2 - b_2 a_{3,1}^2)$ .

Let us put  $a_{6,2} \equiv 0$ . On the intersection with the locus  $P = 0$ ,  $b_{4,1}^2 - b_2 a_{3,1}^2$ , factorizes to  $(b_{4,1} - a_1 a_{3,1})(b_{4,1} + a_1 a_{3,1})$ . This means that the matter curve  $C_2$  splits into two curves in  $B_3$ :  $C_2 = C_{2,+} \cup C_{2,-}$ . Correspondingly we expect two separate **2s** of  $SU(2)$ .

We can go further, following the  $Sp(1)$  case, and resolve the singularities by one resolution

along  $P = 0$  and one small resolution on the surviving singularity. We end up with the equations:

$$\tilde{X}_4^s: \begin{cases} \begin{pmatrix} X(vX + \frac{1}{4}(a_1^2 + 4a_{2,1}s v)z^2) + \frac{1}{2}b_{4,1}s z^4 & -Y_- \\ -Y_+ & X \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ v s = P \end{cases} \subset X_7. \quad (4.105)$$

Consider the locus  $\{v = 0, b_{4,1} = a_1 a_{3,1}\}$ : The determinant of the  $2 \times 2$  matrix in (4.105) factorizes:

$$\det \begin{pmatrix} \frac{1}{4}a_1^2 z^2 X + \frac{1}{2}a_1 a_{3,1} s z^4 & -y + \frac{1}{2}a_{3,1} s z^3 \\ -y - \frac{1}{2}a_{3,1} s z^3 & X \end{pmatrix} = (y - \Upsilon_{2,+}^s)(y + \Upsilon_{2,+}^s) \quad (4.106)$$

where  $\Upsilon_{2,+}^s = \frac{1}{2}a_{3,1} s z^3 + \frac{1}{2}a_1 z X$ . On each of the two branches, the two lines of the matrix become dependent and then give only one more relation. Hence we get the following two four-cycles in  $X_7$  which are also inside  $\tilde{X}_4^s$ :

$$\hat{C}_{2,s+}^{(D \rightarrow S_1)} = \{v = 0, b_{4,1} = a_1 a_{3,1}, y = \Upsilon_{2,+}^s, X\lambda_2 = (\frac{1}{2}a_1 z X + 2z^3 a_{3,1} s)\lambda_1, v s = P, \}, \quad (4.107)$$

$$\hat{C}_{2,s+}^{(D \leftarrow S_2)} = \{v = 0, b_{4,1} = a_1 a_{3,1}, y = -\Upsilon_{2,+}^s, \lambda_2 = -\frac{1}{2}a_1 z \lambda_1, v \sigma = P, \}.$$

These two four-cycles are in different homology classes. Let us explain their origin. On top of the curve  $C_{2,+}$  of the base  $B_3$  the fiber develops two  $\mathbb{P}^1$ s after the resolution. The states that make up **2** of  $SU(2)$  come from M2-branes wrapped on these  $\mathbb{P}^1$ s. If we fiber the two  $\mathbb{P}^1$ s separately over  $C_{2,+}$ , we get the two different four-cycles in (4.107). We have denoted the branches of the resolved  $SU(2)$  singularity by  $S_1$  and  $S_2$  and the remaining brane by  $D$ .

We claim that the orientation of the M2-branes in a **2** are oriented in an opposite way. To count the number of **2**s given by a flux on the curve  $C_{2,+}$ , we can either integrate  $G_4^{(I_1)}$  on  $\hat{C}_{2,s+}^{(D \rightarrow S_1)}$  or  $-\hat{C}_{2,s+}^{(D \leftarrow S_2)}$ , as the two integrals give the same result.

We find

$$\int_{\hat{C}_{2,s+}^{(D \rightarrow S_1)}} G_4^{(I_1)} = \frac{1}{2} \int_{B_3} [P] \cdot (4\bar{K} - [P]) \cdot [\alpha] = \int_{C_{2,+}} \frac{[\alpha]}{2} \quad (4.108)$$

i.e. we integrate the flux on the brane-imagebrane system on the intersection curve.

### Orientation of the M2-branes

Let us consider one of the curves where the  $SU(2)$  singularity locus intersects the brane-imagebrane system. After the resolution, the fiber develops an extra  $\mathbb{P}^1$  on top of the  $SU(2)$  locus. This  $\mathbb{P}^1$  separates the two singular points of the fiber that were on top of each other before the resolution, creating the singularity of the full space. The M2-brane wrapped on this  $\mathbb{P}^1$  stretches between these two points and gives the degrees of freedom to construct the adjoint of  $SU(2)$ . On top of the matter curve, this  $\mathbb{P}^1$  splits into two  $\mathbb{P}^1$ s:

$$S_1 \longrightarrow S_2 \quad \rightsquigarrow \quad S_1 \longrightarrow D \longrightarrow S_2 \quad (4.109)$$

The arrows stand for oriented  $\mathbb{P}^1$ s. The M2-branes wrapping the sum of the two  $\mathbb{P}^1$ s are still the degrees of freedom of the adjoint of  $SU(2)$ . The M2-branes wrapping the separate  $\mathbb{P}^1$ s make

up a doublet representation of  $SU(2)$ . The charges of the M2-branes with respect to the Cartan of  $SU(2)$  are given by the intersections with the sum of the two  $\mathbb{P}^1$ s (related to the gauge field coming from  $C_3$  along the corresponding two-form). Hence a doublet of  $SU(2)$  is made up of two complex states with opposite charges with respect to the Cartan generator. That is why a doublet is constructed from the states of an M2-brane wrapped on  $\mathbb{P}_{S_1 \rightarrow D}^1$  and another M2-brane wrapped on  $\mathbb{P}_{D \rightarrow S_2}^1$  with the opposite orientation.

## 5 The type IIB picture

In this section we want to check our F-theory results by comparing them with the ones obtained in perturbative type IIB string theory.

For compactifications of F-theory, there exists a region of the moduli space where the string coupling is weak everywhere over  $B_3$ . In this region, one can define a CY threefold that is a double cover of  $B_3$ . If this CY threefold is smooth, we can trust perturbative type IIB string theory and expect the results to match those obtained from F-theory. In the case where the CY threefold is singular, this match is more difficult (see [5]).

The weak coupling region of the CY fourfold moduli space is reached in the following way: One scales the sections  $a_i$  as (see [5, 13])

$$a_3 \rightarrow \epsilon a_3 \quad a_4 \rightarrow \epsilon a_4 \quad a_6 \rightarrow \epsilon^2 a_6, \quad (5.1)$$

where  $\epsilon$  is a complex constant that drives the weak coupling limit. With these redefinitions, the sections  $b_i$  scale like

$$b_2 \rightarrow \epsilon b_2 \quad b_4 \rightarrow \epsilon b_4 \quad b_6 \rightarrow \epsilon^2 b_6. \quad (5.2)$$

At leading order in  $\epsilon \rightarrow 0$ , the string coupling  $g_s = (\text{Im}\tau)^{-1}$  becomes weak everywhere (except near  $b_2 = 0$ ), while the discriminant locus of the elliptic fibration becomes:

$$\Delta \approx \epsilon^2 b_2^2 (b_4^2 - b_2 b_6) + \mathcal{O}(\epsilon^2). \quad (5.3)$$

We see that the 7-brane locus factorizes in two pieces in weak coupling limit. It turns out that the monodromies of the axion-dilaton  $\tau$  around the two separate loci are those of an  $O7$ -plane and a  $D7$ -brane, respectively. We can identify

$$O7 : \quad b_2 = 0 \quad \quad D7 : \quad b_4^2 - b_2 b_6 = 0. \quad (5.4)$$

If the polynomials  $b_i$  are generic, the  $D7$ -brane is one connected surface. Due to the specific structure of its defining equation (5.4), it is not a completely generic divisor [79]. Furthermore, a  $D7$ -brane of the form (5.4) has singular points contained in its worldvolume, around which it has the shape of the so-called Whitney umbrella [70]. We hence refer to a  $D7$ -brane described by an equation of the type (5.4) as a *Whitney type  $D7$ -brane*.

The weak-coupling limit of F-theory can be matched with perturbative type IIB string theory compactified on an orientifold of the CY threefold  $X_3$ :

$$\xi^2 = b_2. \quad (5.5)$$



Here, we have added a new coordinate  $\xi$ . The orientifold involution acts on  $X_3$  by sending  $\xi \mapsto -\xi$ , so that  $X_3$  is a double cover of  $B_3$ , branched over the locus  $\xi = 0$ . The projection map

$$\pi : X_3 \mapsto B_3 \quad (5.6)$$

is 2-1 everywhere except at the branch loci (we assume that there are no  $O3$ -planes). We have the following relation for the Poincaré dual of the divisor class of the  $O7$ -plane:

$$[O7] = \pi^*(\bar{K}) = \pi^*(-c_1(B_3)). \quad (5.7)$$

To avoid cluttering our formulae, we will use the following notation:

$$[O7] = -K \in H_4(X_3, \mathbb{Z}), \quad c_1(B_3) = \bar{K}, \quad \pi^*(K) = K, \quad \pi^*(c_1(B_3)) = -K \in H^2(X_3, \mathbb{Z}). \quad (5.8)$$

We are dropping pull-back symbols and, as before, the distinction between a line bundle, its first Chern class, and its associated divisor.

A suitable technique to describe such a background is based on Sen's tachyon condensation [70, 84, 85]. The idea is to describe the fluxed  $D7$ -brane configuration by using a set of  $D9$ -branes and image anti- $D9$ -branes with suitable vector bundles that eventually condense to the wanted  $D7$ -brane configuration.

### 5.1 Flux on a Whitney-type brane

We will now study the flux on a generic *Whitney-type*  $D7$ -brane in perturbative type IIB theory.

As explained in [70], an orientifold invariant  $D7$ -brane is best described as a tachyon condensate of two  $D9$ -branes and two anti- $D9$ -branes with the following bundles on them:

$$\begin{array}{ccc} \overline{D9_1} & \overline{D9_2} & D9_1 \quad D9_2 \\ \mathcal{L}_a^{-1} \oplus \mathcal{L}_b^{-1} & \xrightarrow{T} & \mathcal{L}_a \oplus \mathcal{L}_b, \end{array}$$

where  $\mathcal{L}_a, \mathcal{L}_b$  are holomorphic line bundles. The most general tachyon matrix respecting the orientifolding has the form:

$$T(\vec{x}, \xi) = \begin{pmatrix} 0 & \eta(\vec{x}) \\ -\eta(\vec{x}) & 0 \end{pmatrix} + \xi \begin{pmatrix} \rho(\vec{x}) & \psi(\vec{x}) \\ \psi(\vec{x}) & \tau(\vec{x}) \end{pmatrix}, \quad (5.9)$$

where  $\eta, \psi, \rho, \tau$  are sections of  $\mathcal{L}_a \otimes \mathcal{L}_b$ ,  $\mathcal{L}_a \otimes \mathcal{L}_b \otimes K$ ,  $\mathcal{L}_a^2 \otimes K$ ,  $\mathcal{L}_b^2 \otimes K$ , respectively. The  $D7$ -brane divisor will be given by

$$\det T = \eta^2 + \xi^2 (\rho \tau - \psi^2) = 0. \quad (5.10)$$

We see that this is the same expression as in (5.4), once we substitute the equation of the CY threefold and identify:

$$\frac{a_3}{2} \equiv \psi, \quad \frac{b_4}{2} \equiv \eta, \quad \frac{b_6}{4} \equiv \psi^2 - \rho \tau. \quad (5.11)$$

Note that the form of  $b_6$  is not generic for non-trivial  $\rho$  and  $\tau$ . As explained in [70], this signals the presence of gauge flux on the  $D7$ -brane which fixes some of the brane moduli. In our construction of  $G_4$  flux, a similar mechanism is at work. In order to construct the flux, we need to fix some

complex structure moduli of the elliptic fourfold such that  $a_6 \equiv \rho\tau$ , i.e. the fourfold takes the form (4.4).

The total charge ‘Mukai’ vector  $\Gamma_{D7}$  of this system is given by the following formula:

$$\begin{aligned}\Gamma_{D7} &= \Gamma_{D9_1} + \Gamma_{D9_2} - \Gamma_{\overline{D9_1}} - \Gamma_{\overline{D9_2}} \\ &= \left( \text{ch}(\mathcal{L}_a) + \text{ch}(\mathcal{L}_b) - \text{ch}(\mathcal{L}_a^{-1}) - \text{ch}(\mathcal{L}_b^{-1}) \right) \left( 1 + \frac{c_2(X_3)}{24} \right) \\ &= 2(c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) + \left[ \frac{1}{3} (c_1(\mathcal{L}_a)^3 + c_1(\mathcal{L}_b)^3) + \frac{1}{12} (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) \cdot c_2(X_3) \right].\end{aligned}\tag{5.12}$$

The first term gives the total  $D7$ -charge and the second one the total  $D3$ -charge, which is induced both by curvature and gauge field-strength on the  $D7$ . Note that all charges in the above expression refer to the physics in the double cover  $X_3$ . The projection to  $B_3$  will hence half the charges.

Since we have only one  $D7$ -brane, its  $D7$ -charge must cancel the  $D7$ -charge of the  $O7$ -plane. The charge vector of the  $O7$ -plane is

$$\Gamma_{O7} = -8[O7] + \frac{\chi([O7])}{6}\omega, \tag{5.13}$$

where  $\omega$  is the volume form on  $X_3$ . The cancellation of the  $D7$ -charge then implies

$$c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b) = -4K = 4\bar{K}. \tag{5.14}$$

For the  $D3$ -charge of the Whitney brane, we can easily disentangle the curvature induced one from the flux induced one. It was argued in [70] that the  $D7$ -brane resulting from the tachyon condensation will have zero gauge flux whenever the bundles on the  $D9$  and anti- $D9$ -branes are such that either  $\rho$  or  $\tau$  are constants. This is equivalent to choosing  $\mathcal{L}_a^2 = K^{-1}$  or  $\mathcal{L}_b^2 = K^{-1}$ . The class of the  $D7$ -brane does not change, i.e.  $[D7] = 2(c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b))$  remains the same class; what changes is the difference  $c_1(\mathcal{L}_a) - c_1(\mathcal{L}_b)$ .

We then substitute  $c_1(\mathcal{L}_a) = -\frac{1}{2}K$  (consequently  $c_1(\mathcal{L}_b) = \frac{1}{2}([D7] + K)$ ) into the tadpole formula above<sup>9</sup>:

$$Q_{D3}^c = \frac{1}{24} \int_{X_3} [D7] \cdot (3K^2 + 3K \cdot [D7] + [D7]^2 + c_2(X_3)). \tag{5.15}$$

We can find the flux induced tadpole by subtracting this result from that in (5.12), yielding:

$$\begin{aligned}Q_{D3}^f &= \frac{1}{8} \int_{X_3} [D7] \cdot (2c_1(\mathcal{L}_a) + K) \cdot (2c_1(\mathcal{L}_a) - [D7] - K) \\ &= -\frac{1}{4} \int_{X_3} (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) \cdot (2c_1(\mathcal{L}_a) + K) \cdot (2c_1(\mathcal{L}_b) + K).\end{aligned}\tag{5.16}$$

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<sup>9</sup>If  $B_3$  is not spin, then  $K$  is not divisible by two. In principle, in order to compute the curvature induced tadpole, one should resolve the Whitney brane, and compute its Euler characteristic. However, it turns out that the formal manipulation applied here still works.

## Match with F-theory

To make the comparison with F-theory, we need to remember that the result in (5.17) is given in the double cover, i.e. it gives the number of  $D3$ -branes needed to cancel the tadpole before taking the orientifold projection. After the projection the number of  $D3$ -branes is halved, yielding the number of physical  $D3$ -branes.

If we substitute the relation (5.14) and  $[\rho] = 2c_1(\mathcal{L}_a) + K$ , so that  $[\tau] = 2c_1(\mathcal{L}_b) + K$ , in (5.16) and divide by two to take into account the orientifolding, we obtain:

$$\begin{aligned} Q_{D3} &= -\frac{1}{2} \int_{X_3} \bar{K} \cdot [\rho] \cdot [\tau] \\ &= - \int_{B_3} c_1(B_3) \cdot [\rho] \cdot [\tau]. \end{aligned} \quad (5.17)$$

Hence we obtain precisely the same result as in the corresponding F-theory computation, (4.27).

## Algebraic definition of the flux

The flux  $G_4$  in F-theory has an algebraic description. It is defined as the difference of two algebraic four-cycles of  $X_4$  that are homologous in the ambient space  $X_5$ , but reside in different homology classes in  $X_4$ . In this section, we want to give an analogous description for the two-form flux on a  $D7$ -brane.

The profile of the relic  $D7$ -brane after the  $D9$ 's and anti- $D9$ 's condense is given by equation (5.10), which has a curve worth of double point singularities at  $\eta = \xi = 0$ .

As described in [70], one can resolve this singularity by blowing up  $X_3$  into a new space  $\hat{X}_3$ . This entails introducing a new coordinate  $t \in \mathcal{L}_a \otimes \mathcal{L}_b \otimes K$  and imposing the constraint  $\eta = t\xi$ . We will not follow this procedure here, as we do not need to make explicit computations with the flux.

The line bundle that lives on the brane corresponds to the ‘quotient’  $E/T(F)$ , where

$$T : F = \mathcal{L}_a^{-1} \oplus \mathcal{L}_b^{-1} \mapsto E = \mathcal{L}_a \oplus \mathcal{L}_b. \quad (5.18)$$

A typical section  $s = (\lambda_a, \lambda_b)$  of the quotient bundle will have its vanishing locus wherever both components are zero, or wherever it lies in the image of  $T$ . Clearly, the system of equations  $\lambda_a = \lambda_b = 0$  will be satisfied along a curve that intersects, but does not lie on the  $D7$ -brane. Consequently, we will neglect this locus.

Define a matrix  $\tilde{T}$  such that  $T\tilde{T} = \det(T)$ :

$$\tilde{T}(\vec{x}, \xi) = \begin{pmatrix} \xi\tau(\vec{x}) & -\eta - \xi\psi(\vec{x}) \\ +\eta - \xi\psi(\vec{x}) & \xi\rho(\vec{x}) \end{pmatrix}. \quad (5.19)$$

Hence  $\tilde{T}$  is constructed such that its kernel is the image of  $T$  away from the points where the rank of  $T$  decreases to 0<sup>10</sup>. The brane wraps the divisor  $S$  described by equation (5.10). Its homology class is  $[S] = 2(c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b))$ .

<sup>10</sup> If  $s$  lies in the image of  $T$ , i.e.  $s = T\tilde{s}$ , for some  $\tilde{s} \in \Gamma(\mathcal{L}_a^{-1} \oplus \mathcal{L}_b^{-1})$ , then

$$\tilde{T}s|_{D7} = \det(T)\tilde{s}|_{D7} = 0. \quad (5.20)$$

The locus on the brane where  $s$  becomes invertible is given by the following intersection locus  $\mathcal{I}$ :

$$\mathcal{I} : \quad \tilde{T} \cdot \begin{pmatrix} \lambda_a \\ \lambda_b \end{pmatrix} = 0 \quad \sim \quad \lambda_a \xi \tau = \lambda_b (\eta + \xi \psi) \quad \cap \quad \lambda_a (\xi \psi - \eta) = \lambda_b \xi \rho. \quad (5.22)$$

This locus contains two branches  $\mathcal{I} = \mathcal{I}_\sigma \cup \mathcal{I}_{ab}$ , where

$$\mathcal{I}_\sigma : \quad \tilde{T} s = 0, s \neq (0, 0), \quad \mathcal{I}_{ab} : \quad s = (\lambda_a, \lambda_b) = (0, 0). \quad (5.23)$$

The total flux  $F_2$  on the brane should correspond to the Poincaré dual to the branch  $\mathcal{I}_\sigma$  plus the usual  $+c_1(S)/2$  shift.

$$F_2 = PD_S(\mathcal{I}_\sigma) + \frac{1}{2} c_1(S). \quad (5.24)$$

This flux  $F_2$  does not induce any  $D5$ -charge, i.e. its push-forward is trivial in the ambient threefold. In fact,

$$\begin{aligned} PD_{X_3}(\mathcal{I}_\sigma) &= PD_{X_3}(\tilde{T} s = 0) - PD_{X_3}(s = (0, 0)) \\ &= (2c_1(\mathcal{L}_b) + c_1(\mathcal{L}_a)) \cdot (2c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) - c_1(\mathcal{L}_a) \cdot c_1(\mathcal{L}_b), \\ &= 2(c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b))^2 = \frac{1}{2}[S] \cdot c_1(S), \end{aligned} \quad (5.25)$$

where we used  $c_1(S) = -[S]$ . Putting all the information together, we find

$$i_*(F_2) = PD_{X_3}(\mathcal{I}_\sigma) + \frac{1}{2} c_1(S) \cdot [S] = 0. \quad (5.26)$$

In particular, this flux respects the involution condition  $\sigma^*(F_2) = -F_2$  [70].

## 5.2 Brane-Imagebrane

Let us consider the case when the Whitney brane splits into a brane and its image. Without loss of generality we restrict to the case when  $2c_1(\mathcal{L}_a) - 4\bar{K} \geq 0$ . We get a brane and its image in the double cover when <sup>11</sup>  $\tau = 0$  and  $\rho = -2\psi\beta$ , where  $\beta$  is a section of  $\mathcal{L}_a \otimes \mathcal{L}_b^{-1}$  [70]. One can fix  $\beta \equiv 0$  by a gauge transformation, and then also put  $\rho \equiv 0$ . In what follows, we will associate the brane/imagebrane case to the deformation that brings both  $\rho \equiv 0$  and  $\tau \equiv 0$ .

The tachyon matrix becomes:

$$T(\vec{x}, \xi) = \begin{pmatrix} 0 & \eta(\vec{x}) + \xi \psi(\vec{x}) \\ -\eta(\vec{x}) + \xi \psi(\vec{x}) & 0 \end{pmatrix}. \quad (5.27)$$

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Conversely, suppose we are at a locus on the  $D7$ , then  $T$  must have rank one. The rank cannot go down to zero since that would require four constraints to be satisfied. Therefore, in some basis it can be rewritten as:

$$T = \xi \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \tilde{T} = \xi \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix} \quad \text{for some } P. \quad (5.21)$$

It follows that the section must lie in the image of  $T$  if it is annihilated by  $\tilde{T}$  outside  $P = 0$  and  $\xi = 0$ . It is clear that this logic only applies on the  $D7$ -brane, as  $T$  is surjective away from there.

<sup>11</sup>We want that  $T$  is deformed such that it can be brought to an off-diagonal form by a well defined gauge transformation (i.e. a gauge transformation that brings a regular field strength to a regular one). This forces one among  $\rho$  and  $\tau$  to be identically zero (depending on which one has smaller degree) and the other one to be proportional to  $\psi$ , so that it takes a factorized form.

The determinant of  $T$  factorizes, so that the  $D7$ -brane splits into two branes which are identified by the orientifold involution:

$$\det T = (\eta + \xi \psi)(\eta - \xi \psi) = 0 \quad \rightarrow \quad S_+ : \eta = \xi \psi \quad S_- : \eta = -\xi \psi. \quad (5.28)$$

Since each of these surfaces is not singular we will not need to blow up.

In this configuration the gauge group is  $U(1)$ . If brane and imagebrane recombine back into a single Whitney brane, the gauge group breaks from  $U(1)$  to  $O(1)$  [67, 70].

This system is still described by the condensate of two  $D9$ -branes and two anti- $D9$ -branes. The action of the tachyon, however, is such that brane and imagebrane are described separately by the following sequences:

$$\begin{array}{ccccccc} \overline{D9_2} & & D9_1 & & \overline{D9_2} & & D9_1 \\ D7: & \mathcal{L}_b^{-1} & \xrightarrow{T} & \mathcal{L}_a & D7': & \mathcal{L}_a^{-1} & \xrightarrow{T} & \mathcal{L}_b \end{array}$$

The ‘Mukai’ vectors of the two branes are

$$\begin{aligned} \Gamma &= (\text{ch}(\mathcal{L}_a) - \text{ch}(\mathcal{L}_b^{-1})) \left( 1 + \frac{c_2(X_3)}{24} \right) \\ &= (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) + (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) \frac{c_1(\mathcal{L}_a) - c_1(\mathcal{L}_b)}{2} + Q_{D3}^{D7}, \\ \Gamma' &= (\text{ch}(\mathcal{L}_b) - \text{ch}(\mathcal{L}_a^{-1})) \left( 1 + \frac{c_2(X_3)}{24} \right) \\ &= (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) + (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) \frac{c_1(\mathcal{L}_b) - c_1(\mathcal{L}_a)}{2} + Q_{D3}^{D7'}. \end{aligned}$$

In both cases, the  $D3$ -charge is given by

$$Q_{D3}^{D7} = Q_{D3}^{D7'} = (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) \cdot \frac{\chi(S_-)}{24} + \frac{1}{2} (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) \left( \frac{c_1(\mathcal{L}_a) - c_1(\mathcal{L}_b)}{2} \right)^2. \quad (5.29)$$

We have used that  $S_+$  and  $S_-$  are homologous and that the Euler characteristic of a divisor  $D$  in a CY threefold is given by  $\chi(D) = D^3 + c_2(X_3) \cdot D$ .

The first term in (5.29) is the geometric contribution, while the second one is the flux  $D3$ -tadpole in the usual form  $Q_{D3}^F = \frac{1}{2} \int_{D7} F_2 \cdot F_2$ .

### Match with F-theory

If we sum the flux contributions of the two  $D7$ -branes, we get

$$Q_{D3}^f = \int_{X_3} (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) \left( \frac{c_1(\mathcal{L}_a) - c_1(\mathcal{L}_b)}{2} \right)^2. \quad (5.30)$$

The relation (5.14) still holds, so that the physical  $D3$ -tadpole after orientifolding is given by

$$Q_{D3}^f = -\frac{1}{2} \int_{X_3} 4K \left( \frac{[\alpha]}{2} \right)^2 = \int_{B_3} c_1(B_3) [\alpha]^2. \quad (5.31)$$

In the above equation,  $\alpha$  is a section of  $\mathcal{L}_a \otimes \mathcal{L}_b^{-1}$ . Hence

$$[\alpha] = c_1(\mathcal{L}_a) - c_1(\mathcal{L}_b) = 2c_1(\mathcal{L}_a) - 4\bar{K} = [\rho] - 3\bar{K}, \quad (5.32)$$

so that we find perfect agreement with the corresponding F-theory result, (4.33) and (4.52).

### Algebraic definition of the flux

Let us see how the flux looks like when the Whitney brane factorizes into a brane and its image. Its Poincaré dual is still given by the locus  $\mathcal{I}_\sigma = \mathcal{I} - \mathcal{I}_{ab}$ . After the transition, the locus  $\mathcal{I}$  is

$$\mathcal{I} : \tilde{T} \cdot \begin{pmatrix} \lambda_a \\ \lambda_b \end{pmatrix} = 0 \quad \rightarrow \quad \begin{cases} (\eta + \xi\psi)\lambda_b = 0 \\ (\eta - \xi\psi)\lambda_a = 0 \end{cases}. \quad (5.33)$$

We note that it splits into four branches:

$$\begin{aligned} \mathcal{I}_{ab} : \lambda_b = 0 \cap \lambda_a = 0 & & \mathcal{I}_0 : \eta = 0 \cap \xi\psi = 0 \\ \mathcal{I}_- : \eta = -\xi\psi \cap \lambda_a & & \mathcal{I}_+ : \eta = \xi\psi \cap \lambda_b = 0. \end{aligned} \quad (5.34)$$

As before, we drop the branch  $\mathcal{I}_{ab}$ . The branch  $\mathcal{I}_0$  also does not contribute as the rank of  $T$  goes to zero there: On the curve  $\eta = 0$  inside the  $D7$ -brane, the quotient bundle  $E/T(F)$  has rank two. Hence a generic section of this bundle does not vanish anywhere on this curve.

The two surviving branches define the flux on  $S_-$  and the flux on  $S_+$ :  $\mathcal{I}_-$  is a curve of class  $c_1(\mathcal{L}_a)$  on  $S_-$ , while  $\mathcal{I}_+$  is a curve of class  $c_1(\mathcal{L}_b)$  on  $S_+$ . To have the flux, we need to add half of the first Chern class of the corresponding surface:  $F_\pm = PD_{S_\pm}(\mathcal{I}_\pm) + \frac{1}{2}c_1(S_\pm)$ . We obtain

$$F_- = \frac{1}{2}(c_1(\mathcal{L}_a) - c_1(\mathcal{L}_b)) \quad F_+ = \frac{1}{2}(c_1(\mathcal{L}_b) - c_1(\mathcal{L}_a)) \quad (5.35)$$

where we used  $c_1(S_\pm) = -[S_\pm]$  and  $[S_+] = [S_-] = c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)$ .

### Chiral states

When the Whitney brane splits into a  $D7$ -brane and its image, we have a background with two  $D7$ -branes. At their intersection we have 6D (massless) matter. In presence of gauge flux on the  $D7$ -branes, the 4D spectrum of matter can become chiral.

Having the Mukai vector, one can compute the number of chiral states. The DSZ intersection product between two charge vectors is given by

$$\langle \Gamma_1, \Gamma_2 \rangle = \int_X \Gamma_1 \Gamma_2^*, \quad (5.36)$$

where the  $\Gamma^*$  is obtained in this case by flipping the two-form component of  $\Gamma$ . It was shown in [70, 86] that the chiral index for the spectrum between a brane and its orientifold image is given by:

$$I_o(\Gamma) = \frac{1}{2} \left( \langle \Gamma', \Gamma \rangle + \frac{1}{4} \langle \Gamma_{O7}, \Gamma \rangle \right), \quad (5.37)$$

where, in our case  $\Gamma_{O7} = 8K + \frac{\chi([O7])}{6}\omega$ . This is an equivariant index that computes the chiral spectrum living at the curve where the  $D7$  meets its image:  $\eta = \psi = 0$ , away from the  $O7$ , as

opposed to the curve  $\eta = \xi = 0$ . In our case, the index gives

$$I_o(D7) = \int_{X_3} (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) \cdot \frac{c_1(\mathcal{L}_a) - c_1(\mathcal{L}_b)}{2} \cdot (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b) + K) . \quad (5.38)$$

$$(5.39)$$

Again, we make use of the relation (5.14), so that we can express the chiral index as

$$I_o(D7) = 12 \int_{B_3} \bar{K}^2 \cdot [\alpha] = 12 \int_{B_3} c_1(B_3)^2 \cdot [\alpha] , \quad (5.40)$$

where  $\alpha$  is a section of  $\mathcal{L}_a \otimes \mathcal{L}_b^{-1}$ . This is the same result we have obtained in F-theory by integrating the corresponding  $G_4$  over the matter surface (4.55).

### 5.3 $Sp(1)$ stack plus a Whitney-type brane

In this section, we consider the weak coupling limit of the configuration discussed in section 4.4.1. In type IIB language, we then have an orientifold-invariant stack of two  $D7$ -branes, which intersect the  $O7$ -plane transversally. Each of them wraps the divisor  $P = 0$ , where  $P$  is a section of the line bundle  $\mathcal{L}_P$ . The remaining  $D7$ -tadpole is saturated by a single Whitney-type  $D7$ -brane.

The corresponding system of  $D9 - \overline{D9}$  is given by

$$\begin{array}{ccccccc} \overline{D9_1} & & \overline{D9_2} & & \overline{D9_3} & & \overline{D9_4} \\ \mathcal{L}_a^{-1} & \oplus & \mathcal{L}_b^{-1} & \oplus & \mathcal{L}_P^{-1/2} \otimes \mathcal{L}_q^{-1/2} & \oplus & \mathcal{L}_P^{-1/2} \otimes \mathcal{L}_q^{1/2} , \\ \\ D9_1 & & D9_2 & & D9_3 & & D9_4 \\ \mathcal{L}_a & \oplus & \mathcal{L}_b & \oplus & \mathcal{L}_P^{1/2} \otimes \mathcal{L}_q^{1/2} & \oplus & \mathcal{L}_P^{1/2} \otimes \mathcal{L}_q^{-1/2} , \end{array}$$

where  $\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_P, \mathcal{L}_q$  are holomorphic line bundles. The divisor class of the  $O7$  is equal to  $\pi^*(\bar{K})$ . The tachyon matrix of this system is given by:

$$T = \left( \begin{array}{cc} \left( \begin{array}{cc} \xi \rho & \eta + \xi \psi \\ -\eta + \xi \psi & \xi \tau \end{array} \right) & \\ & \left( \begin{array}{cc} 0 & P \\ -P & 0 \end{array} \right) \end{array} \right) , \quad (5.41)$$

where  $\eta, \psi, \rho, \tau, P$  are sections of  $\mathcal{L}_a \otimes \mathcal{L}_b$ ,  $\mathcal{L}_a \otimes \mathcal{L}_b \otimes K$ ,  $\mathcal{L}_a^2 \otimes K$ ,  $\mathcal{L}_b^2 \otimes K$ ,  $\mathcal{L}_P$ , respectively.

The  $D7$  locus has two factors:

$$\det T = P^2(\eta^2 + \xi^2(\rho\tau - \psi^2)) = 0 \quad \rightarrow \quad S_W : \eta^2 + \xi^2(\rho\tau - \psi^2) = 0 \quad S_S : P = 0. \quad (5.42)$$

We see that this is the same result that we obtain by taking (5.4) and substituting the equation of the CY threefold ( $b_2 = \xi^2$ ) and

$$\frac{a_3}{2} \equiv \psi \cdot P \quad \frac{b_4}{2} \equiv \eta \cdot P \quad \frac{b_6}{4} \equiv (\psi^2 - \rho\tau) \cdot P^2 . \quad (5.43)$$



If we have zero flux on the branes, the gauge group is  $O(1) \times Sp(1)$ , where the first factor come from the Whitney brane, and the second one from the invariant stack of two branes<sup>12</sup>.

In the following, we denote the Whitney-type brane by  $W$  and the two branes on the  $Sp(1)$  stack by  $S$  and  $S'$ . The charge vectors are:

$$\begin{aligned}\Gamma_W &= (\text{ch}(\mathcal{L}_a) + \text{ch}(\mathcal{L}_b) - \text{ch}(\mathcal{L}_a^{-1}) - \text{ch}(\mathcal{L}_b^{-1})) \left(1 + \frac{c_2(X_3)}{2}\right) \\ &= 2c_1(\mathcal{L}_a) + 2c_1(\mathcal{L}_b) + \left[\frac{1}{3}(c_1(\mathcal{L}_a)^3 + c_1(\mathcal{L}_b)^3) + \frac{1}{12}(c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b))c_2(X_3)\right]\end{aligned}\quad (5.44)$$

$$\begin{aligned}\Gamma_S &= \text{ch}(\mathcal{L}_P^{1/2} \otimes \mathcal{L}_q^{1/2}) - \text{ch}(\mathcal{L}_P^{-1/2} \otimes \mathcal{L}_q^{1/2}) \\ &= c_1(\mathcal{L}_P) + c_1(\mathcal{L}_P) \frac{c_1(\mathcal{L}_q)}{2} + \left[\frac{1}{2}c_1(\mathcal{L}_P) \left(\frac{c_1(\mathcal{L}_q)}{2}\right)^2 + \frac{1}{24}(c_1(\mathcal{L}_P)^3 + c_2(X_3)c_1(\mathcal{L}_P))\right]\end{aligned}\quad (5.45)$$

$$\begin{aligned}\Gamma'_S &= \text{ch}(\mathcal{L}_P^{1/2} \otimes \mathcal{L}_q^{-1/2}) - \text{ch}(\mathcal{L}_P^{-1/2} \otimes \mathcal{L}_q^{-1/2}) \\ &= c_1(\mathcal{L}_P) - c_1(\mathcal{L}_P) \frac{c_1(\mathcal{L}_q)}{2} + \left[\frac{1}{2}c_1(\mathcal{L}_P) \left(-\frac{c_1(\mathcal{L}_q)}{2}\right)^2 + \frac{1}{24}(c_1(\mathcal{L}_P)^3 + c_2(X_3)c_1(\mathcal{L}_P))\right].\end{aligned}\quad (5.46)$$

We see that the flux on the  $Sp(1)$  stack is given by  $F_S = \frac{c_1(\mathcal{L}_q)}{2}$ . The cancellation of Freed-Witten anomalies says that we must choose  $\mathcal{L}_q$  such that the difference  $c_1(\mathcal{L}_P) - c_1(\mathcal{L}_q)$  is even.

Depending on the choice of  $\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_q$ , we can have a non-zero flux on all the branes. We will consider the case in which we have a flux on the  $Sp(1)$  stack, as this is the only flux that can generate chiral states on the intersections of the  $D7$ -branes. As one can see from the charge vector, the flux is proportional to the zero locus of a section of  $\mathcal{L}_q$  in this case. Such a flux breaks the gauge group  $Sp(1)$  to the  $U(1)$  generated by the Cartan element.

From the charge vectors we can read off the  $D3$ -tadpole of the flux in the same way as in the previous sections. For the Whitney brane, we have exactly the same result as before:

$$Q_{D3}^W = -\frac{1}{4} \int_{X_3} (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) \cdot (2c_1(\mathcal{L}_a) + K) \cdot (2c_1(\mathcal{L}_b) + K), \quad (5.47)$$

where now  $c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b) = 4\bar{K} - c_1(\mathcal{L}_P)$ . As before, we pull the integral down to the quotient  $B_3$  and divide the charges we have computed before orientifolding by 2. Our result is

$$Q_{D3}^W = - \int_{B_3} (\bar{K} - \frac{1}{4}[P]) \cdot [\rho] \cdot (6\bar{K} - [\rho] - 2[P]). \quad (5.48)$$

which matches with the  $D3$ -charge of the flux  $G_4^{(I_1)}$  obtained from F-theory (4.70).

Both branes of the  $Sp(1)$  stack contribute the same  $D3$ -tadpole. After orientifolding it is given by

$$Q_{D3} = Q_{D3}^S + Q_{D3}^{S'} = \frac{1}{2} \int_{X_3} c_1(\mathcal{L}_P) \cdot \left(\frac{c_1(\mathcal{L}_q)}{2}\right)^2 = \int_{B_3} [P] \cdot \left(\frac{[q]}{2}\right)^2. \quad (5.49)$$

This matches with the charge of  $G_4^{(\text{Sp})}$  in (4.73).

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<sup>12</sup>For  $2n$  branes transverse to the  $O7$ -plane, the gauge group is  $Sp(n)$ ; in our case we have  $Sp(1) \cong SU(2)$ .

## Chiral states

After switching on the flux on the  $Sp(1)$  stack, the unbroken gauge group is  $U(1)$ <sup>13</sup>. The flux along this  $U(1)$  allows for a net number of chiral states living at the intersection with the Whitney brane.

The number of chiral states stretching between one brane on  $P = 0$  with flux  $F_S = \frac{c_1(\mathcal{L}_q)}{2}$  and the Whitney brane is given by

$$\begin{aligned}\langle \Gamma_S, \Gamma_W \rangle &= \int_{X_3} \Gamma_S \Gamma_W^* = - \int_{[W] \cap [P]} F_S \\ &= - \int_{X_3} (2c_1(\mathcal{L}_a) + 2c_1(\mathcal{L}_b)) \cdot c_1(\mathcal{L}_P) \cdot \frac{c_1(\mathcal{L}_q)}{2} .\end{aligned}\quad (5.50)$$

The intersection of the second brane in the stack with the Whitney brane is the orientifold image of the one considered. As the Whitney brane is an invariant brane and the orientifold transformation changes the orientation of the strings, the strings  $S \rightarrow W$  are mapped to the strings  $S' \leftarrow W$ . In fact, as expected, the number of chiral states is the same in the two intersections:  $\langle \Gamma_W, \Gamma'_S \rangle = \langle \Gamma_S, \Gamma_W \rangle$ .

Using the fact that  $c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b) = 4\bar{K} - [P]$  and that  $\bar{K} = c_1(B_3)$ , we obtain

$$\langle \Gamma_S, \Gamma_W \rangle = - \int_{B_3} (8c_1(B_3) - 2[P]) \cdot [P] \cdot [q] . \quad (5.51)$$

This is the same chiral index we have computed in F-theory by integrating the flux  $G_4^{(\text{Sp})}$  along the curve  $\hat{C}$ , (4.75).

Other chiral states come from the bulk of the  $Sp(1)$  stack. In general, the number of such states is given by [70]

$$\begin{aligned}\frac{1}{2} (\langle \Gamma'_{D7}, \Gamma_{D7} \rangle + \frac{1}{4} \langle \Gamma_{O7}, \Gamma_{D7} \rangle) &= - \int_{X_3} [D7] \cdot F \cdot (c_1([D7]) - K) \\ &= - \int_{[D7]} F \cdot (c_1([D7]) - K) ,\end{aligned}$$

where  $\Gamma_{O7} = 8K + \frac{\chi([O7])}{6}\omega$ .

In our case we obtain

$$\begin{aligned}\frac{1}{2} (\langle \Gamma'_S, \Gamma_S \rangle + \frac{1}{4} \langle \Gamma_{O7}, \Gamma_S \rangle) &= \int_{X_3} c_1(\mathcal{L}_P) \cdot \frac{c_1(\mathcal{L}_q)}{2} \cdot (c_1(\mathcal{L}_P) + K) \\ &= \int_{B_3} [P] \cdot [q] \cdot ([P] - \bar{K}) .\end{aligned}$$

i.e. the result we obtained in F-theory, using the conjectured formula (4.77).

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<sup>13</sup>We do not write the irrelevant  $O(1)$  factor.

## 5.4 $Sp(1)$ singularity plus brane/imagebrane

In this section we split the Whitney brane of the last section into separate brane and imagebrane while keeping the  $Sp(1)$  stack. We can obtain this configuration by considering a tachyon matrix (5.41) with  $\tau \equiv 0 \equiv \rho$ . In this case, the determinant of  $T$  factorizes into three branches:

$$\det T = P^2(\eta^2 - \xi^2\psi^2) = 0 \quad \rightarrow \quad S_D : \eta = \xi\psi \quad S_{D'} : \eta = -\xi\psi \quad S_S : P = 0. \quad (5.52)$$

Again, we denote the two branes on the  $Sp(1)$  stack by  $S$  and  $S'$ . The two  $D7$ -branes coming from the Whitney one are now called  $D$  and  $D'$ . The charge vectors of the  $D7$ -brane and its image are

$$\begin{aligned} \Gamma_D &= \text{ch}(\mathcal{L}_a) - \text{ch}(\mathcal{L}_b^{-1}) \\ &= (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) + (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) \frac{c_1(\mathcal{L}_a) - c_1(\mathcal{L}_b)}{2} + Q_{D3}^{(D)} \\ \Gamma_{D'} &= \text{ch}(\mathcal{L}_b) - \text{ch}(\mathcal{L}_a^{-1}) \\ &= (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) + (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) \frac{c_1(\mathcal{L}_b) - c_1(\mathcal{L}_a)}{2} + Q_{D3}^{(D')}. \end{aligned} \quad (5.53)$$

$\Gamma_S$  and  $\Gamma'_S$  remain the same as before (see eq. (5.45) and (5.46)).

The  $D3$ -charge of the flux on the  $Sp(1)$  stack is the same as computed in (5.49). The  $D3$ -tadpole of the remaining branes can be read from the charge vectors (5.53). It is equal to (5.30), where now  $c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b) = 4\bar{K} - c_1(\mathcal{L}_P)$ . After orientifolding we find

$$\begin{aligned} Q_{D3}^{(D)} &= \frac{1}{2} \int_{X_3} (4\bar{K} - c_1(\mathcal{L}_P)) \left( \frac{c_1(\mathcal{L}_a) - c_1(\mathcal{L}_b)}{2} \right)^2 \\ &= \int_{B_3} (4c_1(B_3) - [P]) \left( \frac{[\alpha]}{2} \right)^2. \end{aligned} \quad (5.54)$$

This matches with the corresponding F-theory result, (4.83).

### Chiral states - Flux preserving the non-abelian gauge group

First, we want to consider the case of a flux which allows for chiral states, but does *not* break the non-abelian part of the gauge group. Hence we should only switch on a flux on the  $D7$  and its image, but not on the  $Sp(1)$  stack. In the case when there is zero flux on the  $Sp(1)$  stack, the number of chiral states coming from strings  $D \rightarrow S$  is:

$$\int_{X_3} \Gamma_S \Gamma_D^* = \int_{X_3} (4\bar{K} - [P]) \cdot [P] \cdot \frac{c_1(\mathcal{L}_a) - c_1(\mathcal{L}_b)}{2}. \quad (5.55)$$

The same result is obtained for  $\int_{X_3} \Gamma'_S \Gamma_D'^*$ , which counts the chirality of strings stretching  $D \rightarrow S'$ . In fact, the chiral states coming from the two different intersections form a **2** of  $Sp(1)$ . The other intersections ( $\Gamma_S \Gamma_D'^*$  and  $\Gamma'_S \Gamma_D^*$ ) are the images of the studied ones under the orientifold involution.

Taking into account that  $[\alpha] = c_1(\mathcal{L}_a) - c_1(\mathcal{L}_b)$  and that  $\bar{K} = c_1(B_3)$ , the expression (5.55) is exactly the same result obtained in F-theory, (4.97).

## Chiral states - Flux along a Cartan element of the non-abelian group

We now allow for a non-zero flux  $F_S = \frac{c_1(\mathcal{L}_q)}{2}$  on the  $Sp(1)$  stack. The overall gauge group is then broken to  $U(1) \times U(1)$ . We have a different number of chiral states on the two different intersections  $\Gamma_S \Gamma_D^*$  (strings  $D \rightarrow S$ ) and  $\Gamma_D \Gamma_{S'}^*$  (strings  $S' \rightarrow D$ ):

$$\begin{aligned} \int_{X_3} \Gamma_S \Gamma_D^* &= \int_{X_3} (4c_1(B_3) - [P]) \cdot [P] \cdot \left( \frac{[\alpha]}{2} - \frac{[q]}{2} \right), \\ \int_{X_3} \Gamma_D \Gamma_{S'}^* &= - \int_{X_3} (4c_1(B_3) - [P]) \cdot [P] \cdot \left( \frac{[\alpha]}{2} + \frac{[q]}{2} \right). \end{aligned} \quad (5.56)$$

We get the same results for the orientifold images of the chiral intersections, i.e. for the strings  $S' \rightarrow D'$  and  $D' \rightarrow S$ . It is again easy to see that the expressions (5.56) match with the F-theory ones, (4.100).

The flux  $F_S$  also induces chiral bulk states on the  $Sp(1)$  stack, as in the previous case. The number of chiral states in the symmetric representation of  $U(1)$  is still be given by (5.52).

## Chiral states from the intersections of brane and imagebrane

Finally, we compute the chirality induced on the intersection between the  $D7$ -brane and its image. Using  $\Gamma_{O7} = 8K + \frac{\chi([O7])}{6}\omega$  we compute

$$\frac{1}{2} (\langle \Gamma_D', \Gamma_D \rangle + \frac{1}{4} \langle \Gamma_{O7}, \Gamma_D \rangle) = \int_{X_3} (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) \cdot \frac{c_1(\mathcal{L}_a) - c_1(\mathcal{L}_b)}{2} \cdot (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b) - \bar{K}) \quad (5.57)$$

$$\begin{aligned} &= \int_{B_3} (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b)) \cdot (c_1(\mathcal{L}_a) - c_1(\mathcal{L}_b)) \cdot (c_1(\mathcal{L}_a) + c_1(\mathcal{L}_b) - \bar{K}) \\ &= \int_{B_3} (4\bar{K} - [P]) [\alpha] (3\bar{K} - [P]) . \end{aligned} \quad (5.58)$$

This index is reproduced on the F-theory side by integrating the flux  $G_4^{(I_1)}$ , (4.80), over the four-cycle  $\hat{C}_1$ , (4.101).

## 6 The Weierstrass equation as a Pfaffian

### 6.1 $G_4$ -flux via vector bundles

The results of this paper rely on the existence of additional holomorphic four-cycles in the CY fourfold that cannot be written as complete intersections of two divisors with the Weierstrass equation, but as three equations in an ambient fivefold that imply the Weierstrass equation. This is only possible if the complex structure moduli of the fourfold are appropriately tuned. We repeat the main structure here for convenience.

If we impose that the Weierstrass model have a factorizable  $a_6 = \rho \tau$ , then we can write it as:

$$W \equiv Y_- Y_+ - z^6 \rho \tau - X Q = 0, \quad (6.1)$$

where

$$Y_{\pm} = y \pm \frac{1}{2} a_3 z^3, \quad (6.2)$$

and the other variables are polynomials defined in appendix A. Then, the crucial new four-cycles are:

$$\begin{aligned} \sigma_{\rho} : \quad & \{Y_- = 0\} \cap \{X = 0\} \cap \{\rho = 0\}, \\ \sigma_{\tau} : \quad & \{Y_- = 0\} \cap \{X = 0\} \cap \{\tau = 0\}. \end{aligned} \quad (6.3)$$

As we have seen in section 5.1, this factorizability of  $a_6$  also has physical significance when we take Sen's weak coupling limit. It creates new holomorphic two-cycles on the divisor where the  $D7$  is wrapped. In terms of the 'Tate model' variables, the divisor is now:

$$b_4^2 + \xi^2 (4\rho\tau - a_3^2) = 0, \quad (6.4)$$

and the new two-cycles are

$$C_{\rho} : \quad \{b_4 = \pm \xi a_3\} \cap \{\rho = 0\} \quad (6.5)$$

$$C_{\tau} : \quad \{b_4 = \pm \xi a_3\} \cap \{\tau = 0\}. \quad (6.6)$$

These two-cycles imply the existence of a flux that restricts the divisor moduli via the superpotential in [87, 88].

In the IIB setting, we can understand this phenomenon of the enhancement of the Picard lattice of the  $D7$  divisor in a more systematic way: The divisor equation (6.4) is the determinant of a two by two matrix that we physically interpret as encoding the tachyon modes between  $D9$ /anti- $D9$  stacks:

$$T = \frac{1}{2} \begin{pmatrix} 2\xi\rho & b_4 + \xi a_3 \\ -b_4 + \xi a_3 & 2\xi\tau \end{pmatrix}. \quad (6.7)$$

This matrix can be understood as a map between two rank-two holomorphic vector bundles  $T : E_2 \rightarrow F_2$ , such that the following short exact sequence of sheaves:

$$0 \rightarrow E_2 \xrightarrow{T} F_2 \rightarrow \mathcal{E}_1 \rightarrow 0 \quad (6.8)$$

defines a sheaf  $\mathcal{E}_1$  that corresponds to a line bundle  $L_1$  with support on the  $D7$  divisor given by:

$$\det(T) = b_4^2 + \xi^2 (4\rho\tau - a_3^2) = 0, \quad (6.9)$$

The holomorphic curves (6.5) are nothing but the vanishing loci of some typical sections of this line bundle. As explained in [70] and section 5.1, a section  $s$  of the sheaf  $\mathcal{E}$  can be written as a section  $s_E$  of  $F_2$  modulo the image  $T(E_2)$ , hence the vanishing locus of  $s$  is simply the locus where

$$s_E = T(s_F) \quad (6.10)$$

for some  $s_F \in \Gamma(F_2)$ . Searching for such loci corresponds to solving the equation:

$$\tilde{T} \cdot \begin{pmatrix} \lambda_a \\ \lambda_b \end{pmatrix} = \begin{pmatrix} 2\xi\tau & -b_4 - \xi a_3 \\ b_4 - \xi a_3 & 2\xi\rho \end{pmatrix} \cdot \begin{pmatrix} \lambda_a \\ \lambda_b \end{pmatrix} = 0. \quad (6.11)$$

in the CY threefold. The solutions will be holomorphic curves in the threefold that are automatically contained in the divisor. The curves in (6.5) are a special case of this, and can be obtained roughly by choosing one of the  $\lambda$ 's to be a constant.

This  $D7$  divisor is what is known in mathematics as a *determinantal variety*. Such varieties have been studied for a long time (see [89, 90]). The point is that whenever a divisor can be written as the determinant of a map, then the divisor will admit a holomorphic line bundle on it defined via the exact sequence (6.8).

The fundamental reason for the presence of such curves is the following: The locus of the  $D7$  is by definition located wherever the determinant of the two by two matrix is less than two. However, unlike a polynomial, a two by two matrix of polynomials can have loci where it has rank that is not maximal, but also not zero. This is where the special curves are located.

Given the similarity of the equations defining the curves on the  $D7$ -brane in (6.5) and the four-cycles (6.3) in our restricted Weierstrass equation, one cannot help but wonder whether the Weierstrass equation might not also admit some structure like that of a determinantal variety. This turns out *not* to be the case.

However, this is good news. If the fourfold *were* determinantal, then it would admit a new class of line bundles, and their corresponding six-cycles. However, we are interested in new holomorphic *four-cycles*. More precisely, we would like the fourfold to admit a holomorphic *rank-two* vector bundle  $V_2$ , such that its second Chern class is related to the flux

$$G_4 = c_2(V_2) - \delta, \quad (6.12)$$

where  $\delta$  is some subtraction term that ensures the Poincaré invariance of  $G_4$ . It will be of the form  $\delta = \omega_1 \wedge \omega_2$ , where the  $\omega$  are two-forms in  $X_4$ . Since the second Chern class must be a quantized  $(2, 2)$ -form, a rank two holomorphic vector bundle would have the right properties to describe a  $G_4$ -flux, modulo the issue of semi-integral quantization.

It turns out that there is a suitable structure available. If the Weierstrass equation can be written as the *Pfaffian of an anti-symmetric matrix*  $M$ , then, in the ambient fivefold  $X_5$ , we can write an exact sequence:

$$0 \rightarrow E_4 \xrightarrow{M} F_4 \rightarrow \mathcal{E}_2 \rightarrow 0, \quad (6.13)$$

where  $E_4, F_4$  are holomorphic vector bundles of rank four on  $X_5$ , and  $\mathcal{E}_2$  is a coherent sheaf corresponding to a rank two holomorphic vector bundle  $V_2$  with support over the CY fourfold. Indeed, our Weierstrass equation can be written as the Pfaffian of the following matrix:

$$M = \begin{pmatrix} 0 & X & \rho z^3 & Y_+ \\ -X & 0 & -Y_- & \tau z^3 \\ -\rho z^3 & Y_- & 0 & Q \\ -Y_+ & -\tau z^3 & -Q & 0 \end{pmatrix}. \quad (6.14)$$

Notice the striking similarities with the IIB picture. If one makes the substitution  $y \mapsto \frac{1}{2} b_4$ , then one can recognize the tachyon matrix (6.7) as being imbedded into  $M$  in the off-diagonal blocks.

The fact that the CY fourfold is a *Pfaffian* variety gives similar properties to those of determinantal varieties. However, a four by four anti-symmetric matrix has two pairs of equal and opposite eigenvalues. Therefore, depending on the locus, it can only have ranks four, two and zero, but not three or one. The CY fourfold is located wherever the rank is less than four. However, special four-cycles will appear wherever its rank is two. They can be found (analogously to the case of the IIB curves) by solving equations of the form:

$$\Theta : \widetilde{M} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = 0, \quad \text{where} \quad \vec{v} \equiv \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \Gamma(F_4) \quad (6.15)$$

and

$$\widetilde{M} \equiv \det(M) \cdot M^{-1} = \begin{pmatrix} 0 & -\tau z^3 & Q & -Y_- \\ \tau z^3 & 0 & -Y_+ & X \\ -Q & Y_+ & 0 & -\rho z^3 \\ Y_- & -X & \rho z^3 & 0 \end{pmatrix}. \quad (6.16)$$

The special four-cycles (6.3) can now be rediscovered through an appropriate choice of  $F_4$ .

## 6.2 D3-tadpole computation

To show the power of this re-formulation of the Weierstrass model that incorporates a rank two vector bundle, let us compute the  $D3$ -charge induced by the flux and compare it to our calculations in (5.16) and (4.27).

In the short exact sequence (6.13), let us choose the following rank four vector bundles:

$$\begin{aligned} E_4 &= (\mathcal{L}_a^{-1} \otimes K^{-2} \otimes F^{-4}) \oplus (\mathcal{L}_a \otimes K^2 \otimes F^{-4}) \oplus (\mathcal{L}_a^{-1} \otimes K^{-2} \otimes F^{-5}) \oplus (\mathcal{L}_a \otimes K^2 \otimes F^{-5}) \\ F_4 &= (\mathcal{L}_a \otimes K^2 \otimes F^{-2}) \oplus (\mathcal{L}_a^{-1} \otimes K^{-2} \otimes F^{-2}) \oplus (\mathcal{L}_a \otimes K^2 \otimes F^{-1}) \oplus (\mathcal{L}_a^{-1} \otimes K^{-2} \otimes F^{-1}). \end{aligned} \quad (6.17)$$

One can easily check that this choice is consistent with the entries of the matrix  $M$  in (6.14).

The short exact sequence of these bundles defines a rank two vector bundle  $V_2$  localized on the  $CY_4$  hypersurface  $X_4$  defined by  $\text{Pfaff}(M) = 0$ , such that  $G_4 = c_2(V_2) - \delta$ . Hence, the flux-induced  $D3$ -tadpole should be equal to

$$Q_{D3}^f = -\frac{1}{2} \int_{X_4} (c_2(V_2) - \delta)^2. \quad (6.18)$$

In order to compute this, we will make use of the *Grothendieck-Riemann-Roch* applied to the push-forward of the inclusion map  $\iota$  of the fourfold into the ambient fivefold  $\iota : X_4 \hookrightarrow X_5$ . The theorem relates the Chern character of  $V_2$  to the Chern character of its push-forward  $\iota_*(V_2) = \mathcal{E}_2$

$$\iota_*(\text{ch}(V_2) \cdot \text{Td}(X_4)) = \text{ch}(\mathcal{E}_2) \cdot \text{Td}(X_5), \quad (6.19)$$

where the push-forward on the lhs acts on forms by Poincaré dualizing them to cycles, then pushing them forward, and then re-dualizing into forms. The Chern character of the sheaf  $\mathcal{E}_2$  is given by

$$\text{ch}(\mathcal{E}_2) = \text{ch}(F_4) - \text{ch}(E_4). \quad (6.20)$$

Since the Chern classes of  $X_4$  are made of forms pulled back from  $X_5$ , we can rewrite this more conveniently as follows:

$$\iota_*(\text{ch}(V_2)) = \text{ch}(\mathcal{E}_2) \cdot \text{Td}(F^6), \quad (6.21)$$

whereby the Weierstrass equation is a section of the line-bundle  $F^6$ .

Substituting our choices in (6.17), we find the following for the push-forward of the Chern character of the bundle  $V_2$

$$\iota_*(\text{ch}(V_2)) = 6 F \cdot \mathcal{I}, \quad (6.22)$$

where

$$\begin{aligned} \mathcal{I} = & 2 + (-2 F^2 + (c_1(\mathcal{L}_a) - 2 \bar{K})^2) \\ & + \left( \frac{11}{6} F^4 + (c_1(\mathcal{L}_a) - 2 \bar{K})^2 \cdot \left( \frac{1}{12} (c_1(\mathcal{L}_a) - 2 \bar{K})^2 - F^2 \right) \right), \end{aligned} \quad (6.23)$$

and  $6 F$  is Poincaré dual to the  $X_4$  hypersurface in  $X_5$ . Since the one-form component of  $\mathcal{I}$  is zero, we deduce that the first Chern character of its preimage is zero, hence

$$\begin{aligned} \iota_*(\text{ch}(V_2)) &= 2 + \iota_*(\text{ch}_2(V_2)) + \iota_*(\text{ch}_4(V_2)) \\ &= 2 - \iota_*(c_2(V_2)) + \frac{1}{12} \iota_*(c_2(V_2)^2). \end{aligned} \quad (6.24)$$

Let us decompose  $c_2(V_2)$  into a part that lies in the kernel of the pushforward map, and a part that is orthogonal to it:

$$c_2(V_2) = (c_2(V_2) - \delta) + \delta \quad (6.25)$$

where  $\delta = \iota^*(\tilde{\delta})$ , is the pullback of some four-form in  $X_5$ , such that

$$\iota_*(c_2(V_2) - \delta) = 0 \quad \text{and} \quad (c_2(V_2) - \delta) \cdot \delta = 0. \quad (6.26)$$

Note that with  $\delta$  defined in this way, the flux  $G_4$  is Poincaré invariant.

Using relations (6.26), we deduce the following:

$$\iota_*(c_2(V_2)^2) = \iota_*((c_2(V_2) - \delta)^2 - \delta^2) = \iota_*((c_2(V_2) - \delta)^2) - \iota_*(\delta^2). \quad (6.27)$$

Therefore, we can compute the flux-induced  $D3$ -tadpole as follows:

$$Q_{D3}^f = -\frac{1}{2} \int_{X_4} G_4^2 = -\frac{1}{2} \int_{X_5} \iota_*((c_2(V_2) - \delta)^2). \quad (6.28)$$

In terms of the  $i$ -form components  $\mathcal{I}_i$  of  $\mathcal{I}$ , this is given by

$$\begin{aligned} Q_{D3}^f &= -\frac{1}{2} \int_{X_5} 6 F \cdot (12 \mathcal{I}_4 - \mathcal{I}_2^2) \\ &= 6 \int_{X_4} F^2 \cdot \bar{K} \cdot (2 c_1(\mathcal{L}_a) - \bar{K}) \cdot (2 c_1(\mathcal{L}_a) - 7 \bar{K}) \\ &= \int_{B_3} c_1(B_3) \cdot (2 c_1(\mathcal{L}_a) - c_1(B_3)) \cdot (2 c_1(\mathcal{L}_a) - 7 c_1(B_3)). \end{aligned} \quad (6.29)$$

as predicted in<sup>14</sup> (4.27).

Hence, as we see, this description of our  $G_4$ -flux in terms of coherent sheaves gives a prescription for computing the induced tadpole that is much simpler than a direct calculation of the self-intersection of the four-form.

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<sup>14</sup>Remember that  $\rho$  is a section of  $\mathcal{L}_a^2 \otimes K$  and  $\tau$  is a section of  $\mathcal{L}_b^2 \otimes K$ . Moreover we have the relation  $\mathcal{L}_a \otimes \mathcal{L}_b = K^{-4}$ .



### 6.3 Matrix Factorizations

In this section, we point out some interesting links between our construction and those that appear in the context of matrix factorization both in algebraic geometry and string theory.

The phenomenon we have observed in this section is known in mathematics. Whenever a hypersurface can be written as the determinant or the Pfaffian of a matrix, then new sheaves appear on it. More generally, given a hypersurface equation  $W = 0$ , if one can find two  $n \times n$  matrices  $M_1$ , and  $M_2$  such that

$$M_1 \cdot M_2 = W \cdot \text{id}_n, \quad (6.30)$$

then there exist special sheaves  $S^{(1)}$  and  $S^{(2)}$  with support on the hypersurface, defined by the sequences on the ambient space:

$$0 \rightarrow E_n^{(1)} \xrightarrow{M_1} F_n^{(1)} \rightarrow S^{(1)} \rightarrow 0 \quad (6.31)$$

$$0 \rightarrow E_n^{(2)} \xrightarrow{M_2} F_n^{(2)} \rightarrow S^{(2)} \rightarrow 0 \quad (6.32)$$

whereby  $E_{1,2}$  and  $F_{1,2}$  are vector bundles of rank  $n$ . In our case,  $M_1$  is the matrix defined in (6.14), and  $M_2$  is the matrix  $\tilde{M}$  defined in (6.16).

In [91] and [92], matrix factorizations are exploited in detail for the class of hypersurfaces given by quadrics in  $\mathbb{P}^N$ . Our CY hypersurface is not in general a quadric per se, however the form (6.1) is a quadric with respect to the polynomials  $Y_{\pm}, \rho, \tau, X$  and  $Q$ , and hence shares similar structures to those described in these articles.

The technology of matrix factorizations has found its incarnation in string theory, in the context of D-branes, through the works [93] and [94]. More specific applications were found, where codimension two objects in CY threefolds, i.e. D2-branes on holomorphic curves, could be described via matrix factorizations, in [95, 96]. Those situations are analogous to ours, since they treat the construction of holomorphic codimension two objects that cannot be written as the intersection of two divisors with the CY hypersurface, but must be written as the intersection of three specially chosen divisors in the ambient space.

Finally, in [97], the matrix factorization techniques are applied to study holomorphic curves in K3 surfaces. In particular, ways for enhancing the Picard group are investigated. In our case, we are interested in enhancing the analogous group  $H_{H^2,2}^{2,2}(X_4) \cap H^4(X_4, \mathbb{Z})$ .

It is curious and suggestive that such techniques, which have proven useful in perturbative string theory, should make an appearance in our F-theory setups. One wonders, whether the K-theoretic treatment of M-theory of [98] is related to our construction, despite the fact that ours lives in a thirteen-dimensional spacetime ambient to the M-theory spacetime, as opposed to their twelve-dimensional one. It would be interesting to pursue this connection further.

## 7 Conclusions and Outlook

In this paper, we introduced a new way of looking at the problem of  $G_4$  fluxes in F-theory. We provide a construction technique that is very direct, as it describes them in terms of Poincaré duals of holomorphic four-cycles. The resulting fluxes are automatically quantized, of  $(2, 2)$ -type, and have one leg on the fiber, as required for 4D Poincaré invariance.

Because of their algebraic description, we can directly compute the induced D3-tadpole as the self-intersection of a four-cycle. In simple setups with intersecting branes, we computed chirality indices by integrating our fluxes on four-cycles obtained from resolutions of the singularities along matter curves.

Whenever a weak coupling limit was available, we were able to map our fluxes to worldvolume fluxes of  $D7$ -branes, and successfully match all of our results on  $D3$ -tadpoles and chirality.

Our construction also shows how our fluxes lift some of the complex structure moduli of the fourfold, much like worldvolume fluxes lift D7 divisor moduli. Recently, there has been a lot of progress in computing flux-induced potentials for compactifications of F-theory on Calabi-Yau fourfolds [99–105]. It would be very interesting to use these methods to explicitly see how our fluxes fix some of the complex structure moduli such that the section  $a_6$  of the Weierstrass model factorizes.

Finally, we found a curious incarnation of the treatment of  $D7$ -branes in terms of coherent sheaves (i.e. tachyon condensation of  $D9$  and anti- $D9$ -branes), in our treatment of CY fourfolds with their fluxes. In this picture, our  $G_4$  fluxes appear as rank two vector bundles on the fourfolds.

Our results are succinctly summarized in section 2.

We believe that the constructions presented in this paper will open up new and interesting ways of studying F-theory models. They will also provide a check for phenomenologically oriented models that rely on the spectral cover construction.

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## A Rewriting the Weierstrass model

It is paramount to our construction to cast the familiar Weierstrass model in a specific form. The ordinary Weierstrass form,

$$y^2 = x^3 + xz^4f + z^6g, \quad (\text{A.1})$$

is related to the Tate form

$$Y^2 + a_1XYZ + a_3YZ^3 = X^3 + a_2X^2Z^2 + a_4XZ^4 + a_6Z^6 \quad (\text{A.2})$$

by shifting the coordinates  $x, y, z$ . The  $a_n$  are sections of the  $n$ th power of the anticanonical bundle, so their divisor class is  $n\bar{K}$ .

Introducing the quantities

$$\begin{aligned} b_2 &= a_1^2 + 4a_2 \\ b_4 &= a_1a_3 + 2a_4 \\ b_6 &= a_3^2 + 4a_6 \end{aligned} \tag{A.3}$$

the relations between the sections  $f_4, g_6$  and the discriminant  $\Delta$  can be written as

$$\begin{aligned} f &= -\frac{1}{48} (b_2^2 - 24b_4) \\ g &= \frac{1}{864} (b_2^3 - 36b_2b_4 + 216b_6) \\ \Delta &= 4f^3 + 27g^2 = \frac{1}{64}b_2^2 (b_2b_6 - b_4^2) - \frac{9}{16}b_2b_4b_6 + \frac{1}{2}b_4^3 + \frac{27}{16}b_6^2. \end{aligned} \tag{A.4}$$

If we use the parametrization (A.3) in the standard Weierstrass model (A.1), we can rewrite it in the following way:

$$\begin{aligned} y^2 &= x^3 - z^4x\frac{1}{48}(b_2^2 - 24b_4) + z^6\frac{1}{864}(b_2^3 - 36b_4b_2 + 216b_6) \\ &= (x - \frac{1}{12}z^2b_2) \left( (x - \frac{1}{12}z^2b_2)(x + \frac{1}{6}z^2b_2) + \frac{1}{2}z^4b_4 \right) + \frac{1}{4}z^6b_6. \end{aligned} \tag{A.5}$$

If we also use that  $b_6 = a_3^2 + 4a_6$  we obtain

$$(y - \frac{1}{2}z^3a_3)(y + \frac{1}{2}z^3a_3) - z^6a_6 = (x - \frac{1}{12}z^2b_2) \left( (x - \frac{1}{12}z^2b_2)(x + \frac{1}{6}z^2b_2) + \frac{1}{2}z^4b_4 \right). \tag{A.6}$$

To flesh out the structure of this expression, we introduce the quantities

$$Y_{\pm} = y \pm \frac{1}{2}z^3a_3 \tag{A.7}$$

$$X = x - \frac{1}{12}z^2b_2 \tag{A.8}$$

$$\begin{aligned} Q &= (x - \frac{1}{12}z^2b_2)(x + \frac{1}{6}z^2b_2) + \frac{1}{2}z^4b_4 \\ &= X(X + \frac{1}{4}z^2b_2) + \frac{1}{2}z^4b_4, \end{aligned} \tag{A.9}$$

so that we can write the Weierstrass model in the simple form

$$Y_-Y_+ - z^6a_6 = XQ. \tag{A.10}$$

This form of the Weierstrass equation is the starting point of our investigations of  $G_4$  fluxes and algebraic cycles.

## B The intersection form of an embedded submanifold

Given a smooth map  $f$  between two spaces,  $f : X \mapsto Y$ , there are induced maps on cohomology and homology:

$$f^* : H^{\bullet}(Y) \mapsto H^{\bullet}(X), \quad f_* : H_{\bullet}(X) \mapsto H_{\bullet}(Y). \tag{B.1}$$

In the present case, we have the embedding  $i : X_4 \hookrightarrow X_A$  of our elliptic Calabi-Yau fourfold  $X_4$  into some ambient space  $X_A$ . We would like to show that the intersection product between two-forms on  $X_4$  vanishes if the dual cycle of one is in the kernel of the pushforward  $i_*\gamma = 0$  (‘trivial in the ambient space’) while the other is in the image of the pullback  $\alpha = i^*\beta$  (‘descends from the ambient space’).

We compute

$$\int_{X_4} \gamma \wedge \alpha = \int_{\gamma} \alpha = \int_{\gamma} i^*\beta = \int_{i_*\gamma} \beta = 0, \quad (\text{B.2})$$

where we have used naturalness of the cap product [106]. As  $i_*\gamma = 0$  holds by assumption, the integral vanishes.

## C The exceptional curves of an $Sp(1)$ singularity

We are interested in the exceptional curve that is present when resolve a fourfold that describes an  $Sp(1)$  brane situated at  $P = 0$ , as well as the remaining 7-brane. The resolution of the singularity over the  $Sp(1)$  brane is described by

$$(y + \frac{1}{2}sa_{3,1}z^3)(y - \frac{1}{2}sa_{3,1}z^3) - s^2a_{6,2}z^6 - X(Xv + \frac{1}{4}b_2z^2) + \frac{1}{2}sb_{4,1}z^4 = 0$$

$$vs = P. \quad (\text{C.1})$$

The weights of the coordinates are

$X$	$y$	$z$	$v$	$s$
0	0	$2K$	0	$[P]$
2	3	1	0	0
1	1	0	-1	1

To make the following expressions simple, we omit the coordinate  $z$  in the following. It can be reinstated at any point.

To find the exceptional curve after the resolution, we consider the proper transform over the locus  $P = 0$ . This means we have two branches,  $v = 0$  and  $s = 0$ . The  $s = 0$  branch is given by

$$y^2 - X^2(Xv + \frac{1}{4}b_2) = 0. \quad (\text{C.2})$$

As explained in [83], this is a  $\mathbb{P}^1$  for any point on  $P = 0$ , which correspond to the exceptional root of the Dynkin diagram of  $Sp(1)$ .

The  $v = 0$  branch is more interesting, (C.1) becomes

$$y^2 - (\frac{1}{4}a_{3,1}^2 + a_{6,2})s^2 - \frac{1}{4}b_2X^2 - X\frac{1}{2}sb_{4,1} = 0. \quad (\text{C.3})$$

For any point in the base, this is a  $\mathbb{P}^1$  embedded into the  $\mathbb{P}^2$  with homogeneous coordinates  $y, x, s$ . It represent the Cartan node of the  $Sp(1)$  stack. As it is embedded by a quartic equation, it wraps the  $\mathbb{P}^1$  in  $\mathbb{P}^2$  twice. The two bits are connected by some ‘throat’, so they have the topology of a sphere. This throat can, however, shrink to produce two separate  $\mathbb{P}^1$ s. This is

exactly what happens over special points in the base: (C.3) factorizes if it is the sum of two squares.

The simplest way to determine over which points in the base this happens is the following: Over the locus where the Cartan  $\mathbb{P}^1$  of  $Sp(1)$  degenerates into two  $\mathbb{P}^1$ s, (C.3), considered as a hypersurface in the  $\mathbb{P}^2$  with homogeneous coordinates  $X, y, \sigma$ , becomes singular. The gradient of (C.3) is

$$\begin{aligned} 2ydy &= 0 \\ \left(\frac{1}{2}Xb_{4,1} + 2s\left(\frac{1}{4}a_{3,1}^2 + a_{6,2}\right)\right) ds &= 0 \\ \left(\frac{1}{2}s b_{4,1} + \frac{1}{2}b_2 X\right) dX &= 0. \end{aligned} \quad (\text{C.4})$$

Note that solving those three equations implies (C.3). The last two equations can be rewritten as

$$\begin{pmatrix} b_{4,1} & a_{3,1}^2 + 4a_{6,2} \\ b_2 & b_{4,1} \end{pmatrix} \begin{pmatrix} X \\ s \end{pmatrix} = 0. \quad (\text{C.5})$$

Hence all three equations in (C.4) can only have a simultaneous solution if

$$\det \begin{pmatrix} b_{4,1} & a_{3,1}^2 + 4a_{6,2} \\ b_2 & b_{4,1} \end{pmatrix} = b_{4,1}^2 - b_2(a_{3,1}^2 + 4a_{6,2}) = 0. \quad (\text{C.6})$$

The singularity occurs at  $y = 0$  and the  $X, \sigma$  that solve (C.5). In summary, we have a  $\mathbb{P}^1$ , which we call  $S$ , over generic points of  $P$ . This  $\mathbb{P}^1$  is split into two  $\mathbb{P}^1$ s over  $b_{4,1}^2 - b_2(a_{3,1}^2 + 4a_{6,2}) = 0$ . Denoting these by  $S_1^a$  and  $S_1^b$ , it must be in homology that  $S = S_1^a + S_1^b$ .

The discriminant locus of the configuration we have considered is

$$\Delta = P^2 \left( \frac{1}{64} b_2^2 (b_2 b_{6,2} - b_{4,1}) + P \left( -\frac{1}{16} b_2 b_{4,1} b_{6,2} + \frac{1}{2} b_{4,1}^3 + P \frac{27}{16} b_{6,2}^2 \right) \right). \quad (\text{C.7})$$

The intersection between the curve  $P = 0$  and the remaining part hence has two branches:

$$b_2^2 (b_2 b_{6,2} - b_{4,1}) = 0. \quad (\text{C.8})$$

As we have seen by resolving the  $Sp(1)$  singularity, only the second branch leads to an enhancement of the singularity. Consequently, there is only charged matter at this branch.

## References

- [1] C. Beasley, J. J. Heckman, and C. Vafa, “GUTs and Exceptional Branes in F-theory - I,” *JHEP* **01** (2009) 058, [arXiv:0802.3391 \[hep-th\]](#).
- [2] C. Beasley, J. J. Heckman, and C. Vafa, “GUTs and Exceptional Branes in F-theory - II: Experimental Predictions,” *JHEP* **01** (2009) 059, [arXiv:0806.0102 \[hep-th\]](#).
- [3] R. Donagi and M. Wijnholt, “Model Building with F-Theory,” [arXiv:0802.2969 \[hep-th\]](#).
- [4] R. Donagi and M. Wijnholt, “Breaking GUT Groups in F-Theory,” [arXiv:0808.2223 \[hep-th\]](#).

- [5] R. Donagi and M. Wijnholt, “Higgs Bundles and UV Completion in F-Theory,” [arXiv:0904.1218](#) [[hep-th](#)].
- [6] H. Hayashi, R. Tatar, Y. Toda, T. Watari, and M. Yamazaki, “New Aspects of Heterotic–F Theory Duality,” *Nucl. Phys.* **B806** (2009) 224–299, [arXiv:0805.1057](#) [[hep-th](#)].
- [7] H. Hayashi, T. Kawano, R. Tatar, and T. Watari, “Codimension-3 Singularities and Yukawa Couplings in F- theory,” *Nucl. Phys.* **B823** (2009) 47–115, [arXiv:0901.4941](#) [[hep-th](#)].
- [8] R. Blumenhagen, V. Braun, T. W. Grimm, and T. Weigand, “GUTs in Type IIB Orientifold Compactifications,” *Nucl. Phys.* **B815** (2009) 1–94, [arXiv:0811.2936](#) [[hep-th](#)].
- [9] R. Blumenhagen, T. W. Grimm, B. Jurke, and T. Weigand, “Global F-theory GUTs,” *Nucl. Phys.* **B829** (2010) 325–369, [arXiv:0908.1784](#) [[hep-th](#)].
- [10] S. B. Giddings, S. Kachru, and J. Polchinski, “Hierarchies from fluxes in string compactifications,” *Phys. Rev.* **D66** (2002) 106006, [arXiv:hep-th/0105097](#).
- [11] K. Dasgupta, G. Rajesh, and S. Sethi, “M theory, orientifolds and G-flux,” *JHEP* **08** (1999) 023, [arXiv:hep-th/9908088](#).
- [12] C. Vafa, “Evidence for F-Theory,” *Nucl. Phys.* **B469** (1996) 403–418, [arXiv:hep-th/9602022](#).
- [13] A. Sen, “F theory and orientifolds,” *Nucl.Phys.* **B475** (1996) 562–578, [arXiv:hep-th/9605150](#) [[hep-th](#)].
- [14] F. Denef, “Les Houches Lectures on Constructing String Vacua,” [arXiv:0803.1194](#) [[hep-th](#)].
- [15] T. Weigand, “Lectures on F-theory compactifications and model building,” *Class. Quant. Grav.* **27** (2010) 214004, [arXiv:1009.3497](#) [[hep-th](#)].
- [16] R. Blumenhagen, “Basics of F-theory from the Type IIB Perspective,” *Fortsch. Phys.* **58** (2010) 820–826, [arXiv:1002.2836](#) [[hep-th](#)].
- [17] A. P. Braun, “F-Theory and the Landscape of Intersecting D7-Branes,” [arXiv:1003.4867](#) [[hep-th](#)].
- [18] O. J. Ganor, “A note on zeroes of superpotentials in F-theory,” *Nucl. Phys.* **B499** (1997) 55–66, [arXiv:hep-th/9612077](#).
- [19] R. Blumenhagen, M. Cvetič, and T. Weigand, “Spacetime instanton corrections in 4D string vacua - the seesaw mechanism for D-brane models,” *Nucl. Phys.* **B771** (2007) 113–142, [arXiv:hep-th/0609191](#).
- [20] L. E. Ibanez and A. M. Uranga, “Neutrino Majorana masses from string theory instanton effects,” *JHEP* **03** (2007) 052, [arXiv:hep-th/0609213](#).
- [21] B. Florea, S. Kachru, J. McGreevy, and N. Saulina, “Stringy Instantons and Quiver Gauge Theories,” *JHEP* **05** (2007) 024, [arXiv:hep-th/0610003](#).

- [22] R. Argurio, M. Bertolini, S. Franco, and S. Kachru, “Metastable vacua and D-branes at the conifold,” *JHEP* **06** (2007) 017, [arXiv:hep-th/0703236](#).
- [23] R. Argurio, M. Bertolini, G. Ferretti, A. Lerda, and C. Petersson, “Stringy Instantons at Orbifold Singularities,” *JHEP* **06** (2007) 067, [arXiv:0704.0262 \[hep-th\]](#).
- [24] R. Blumenhagen, S. Moster, and E. Plauschinn, “Moduli Stabilisation versus Chirality for MSSM like Type IIB Orientifolds,” *JHEP* **01** (2008) 058, [arXiv:0711.3389 \[hep-th\]](#).
- [25] R. Blumenhagen, A. Collinucci, and B. Jurke, “On Instanton Effects in F-theory,” *JHEP* **08** (2010) 079, [arXiv:1002.1894 \[hep-th\]](#).
- [26] M. Cvetič, J. Halverson, and I. Garcia-Etxebarria, “Three Looks at Instantons in F-theory – New Insights from Anomaly Inflow, String Junctions and Heterotic Duality,” [arXiv:1107.2388 \[hep-th\]](#).
- [27] M. Bianchi, A. Collinucci, and L. Martucci, “Magnetized E3-brane instantons in F-theory,” [arXiv:1107.3732 \[hep-th\]](#).
- [28] K. Becker and M. Becker, “M-Theory on Eight-Manifolds,” *Nucl. Phys.* **B477** (1996) 155–167, [arXiv:hep-th/9605053](#).
- [29] M. Haack and J. Louis, “M-theory compactified on Calabi-Yau fourfolds with background flux,” *Phys. Lett.* **B507** (2001) 296–304, [arXiv:hep-th/0103068](#).
- [30] T. W. Grimm, “The N=1 effective action of F-theory compactifications,” *Nucl. Phys.* **B845** (2011) 48–92, [arXiv:1008.4133 \[hep-th\]](#).
- [31] R. Valandro, “Type IIB Flux Vacua from M-theory via F-theory,” *JHEP* **03** (2009) 122, [arXiv:0811.2873 \[hep-th\]](#).
- [32] A. P. Braun, A. Hebecker, C. Ludeling, and R. Valandro, “Fixing D7 Brane Positions by F-Theory Fluxes,” *Nucl. Phys.* **B815** (2009) 256–287, [arXiv:0811.2416 \[hep-th\]](#).
- [33] S. H. Katz and C. Vafa, “Matter from geometry,” *Nucl. Phys.* **B497** (1997) 146–154, [arXiv:hep-th/9606086](#).
- [34] J. J. Heckman, J. Marsano, N. Saulina, S. Schafer-Nameki, and C. Vafa, “Instantons and SUSY breaking in F-theory,” [arXiv:0808.1286 \[hep-th\]](#).
- [35] J. J. Heckman and C. Vafa, “Flavor Hierarchy From F-theory,” *Nucl. Phys.* **B837** (2010) 137–151, [arXiv:0811.2417 \[hep-th\]](#).
- [36] J. J. Heckman, A. Tavanfar, and C. Vafa, “The Point of  $E_8$  in F-theory GUTs,” *JHEP* **08** (2010) 040, [arXiv:0906.0581 \[hep-th\]](#).
- [37] J. Marsano, N. Saulina, and S. Schafer-Nameki, “Monodromies, Fluxes, and Compact Three-Generation F-theory GUTs,” *JHEP* **08** (2009) 046, [arXiv:0906.4672 \[hep-th\]](#).
- [38] S. Cecotti, M. C. N. Cheng, J. J. Heckman, and C. Vafa, “Yukawa Couplings in F-theory and Non-Commutative Geometry,” [arXiv:0910.0477 \[hep-th\]](#).



- [39] H. Hayashi, T. Kawano, Y. Tsuchiya, and T. Watari, “Flavor Structure in F-theory Compactifications,” *JHEP* **08** (2010) 036, [arXiv:0910.2762 \[hep-th\]](#).
- [40] R. Tatar, Y. Tsuchiya, and T. Watari, “Right-handed Neutrinos in F-theory Compactifications,” *Nucl. Phys.* **B823** (2009) 1–46, [arXiv:0905.2289 \[hep-th\]](#).
- [41] H. Hayashi, T. Kawano, Y. Tsuchiya, and T. Watari, “More on Dimension-4 Proton Decay Problem in F-theory – Spectral Surface, Discriminant Locus and Monodromy,” *Nucl. Phys.* **B840** (2010) 304–348, [arXiv:1004.3870 \[hep-th\]](#).
- [42] E. Dudas and E. Palti, “Froggatt-Nielsen models from  $E_8$  in F-theory GUTs,” *JHEP* **01** (2010) 127, [arXiv:0912.0853 \[hep-th\]](#).
- [43] E. Dudas and E. Palti, “On hypercharge flux and exotics in F-theory GUTs,” *JHEP* **09** (2010) 013, [arXiv:1007.1297 \[hep-ph\]](#).
- [44] J. P. Conlon and E. Palti, “Aspects of Flavour and Supersymmetry in F-theory GUTs,” *JHEP* **01** (2010) 029, [arXiv:0910.2413 \[hep-th\]](#).
- [45] A. Font and L. E. Ibanez, “Yukawa Structure from  $U(1)$  Fluxes in F-theory Grand Unification,” *JHEP* **02** (2009) 016, [arXiv:0811.2157 \[hep-th\]](#).
- [46] A. Font and L. E. Ibanez, “Matter wave functions and Yukawa couplings in F-theory Grand Unification,” *JHEP* **09** (2009) 036, [arXiv:0907.4895 \[hep-th\]](#).
- [47] L. Aparicio, A. Font, L. E. Ibanez, and F. Marchesano, “Flux and Instanton Effects in Local F-theory Models and Hierarchical Fermion Masses,” [arXiv:1104.2609 \[hep-th\]](#).
- [48] C.-M. Chen and Y.-C. Chung, “Flipped  $SU(5)$  GUTs from  $E_8$  Singularities in F-theory,” *JHEP* **03** (2011) 049, [arXiv:1005.5728 \[hep-th\]](#).
- [49] M. J. Dolan, J. Marsano, N. Saulina, and S. Schafer-Nameki, “F-theory GUTs with  $U(1)$  Symmetries: Generalities and Survey,” [arXiv:1102.0290 \[hep-th\]](#).
- [50] V. K. Oikonomou, “F-theory and the Witten Index,” *Nucl. Phys.* **B850** (2011) 273–286, [arXiv:1103.1289 \[hep-th\]](#).
- [51] C. Cordova, “Decoupling Gravity in F-Theory,” [arXiv:0910.2955 \[hep-th\]](#).
- [52] C. Ludeling, H. P. Nilles, and C. C. Stephan, “The Potential Fate of Local Model Building,” *Phys. Rev.* **D83** (2011) 086008, [arXiv:1101.3346 \[hep-th\]](#).
- [53] J. Marsano, N. Saulina, and S. Schafer-Nameki, “F-theory Compactifications for Supersymmetric GUTs,” *JHEP* **08** (2009) 030, [arXiv:0904.3932 \[hep-th\]](#).
- [54] J. Marsano, N. Saulina, and S. Schafer-Nameki, “Compact F-theory GUTs with  $U(1)_{PQ}$ ,” *JHEP* **04** (2010) 095, [arXiv:0912.0272 \[hep-th\]](#).
- [55] T. W. Grimm, S. Krause, and T. Weigand, “F-Theory GUT Vacua on Compact Calabi-Yau Fourfolds,” *JHEP* **07** (2010) 037, [arXiv:0912.3524 \[hep-th\]](#).
- [56] C.-M. Chen, J. Knapp, M. Kreuzer, and C. Mayrhofer, “Global  $SO(10)$  F-theory GUTs,” *JHEP* **10** (2010) 057, [arXiv:1005.5735 \[hep-th\]](#).



- [57] Y.-C. Chung, “On Global Flipped SU(5) GUTs in F-theory,” *JHEP* **03** (2011) 126, [arXiv:1008.2506 \[hep-th\]](#).
- [58] C.-M. Chen and Y.-C. Chung, “On F-theory  $E_6$  GUTs,” *JHEP* **03** (2011) 129, [arXiv:1010.5536 \[hep-th\]](#).
- [59] M. Cvetič, I. Garcia-Etxebarria, and J. Halverson, “Global F-theory Models: Instantons and Gauge Dynamics,” *JHEP* **01** (2011) 073, [arXiv:1003.5337 \[hep-th\]](#).
- [60] J. Knapp, M. Kreuzer, C. Mayrhofer, and N.-O. Walliser, “Toric Construction of Global F-Theory GUTs,” *JHEP* **03** (2011) 138, [arXiv:1101.4908 \[hep-th\]](#).
- [61] J. Knapp and M. Kreuzer, “Toric Methods in F-theory Model Building,” [arXiv:1103.3358 \[hep-th\]](#).
- [62] R. Friedman, J. W. Morgan, and E. Witten, “Vector bundles over elliptic fibrations,” [arXiv:alg-geom/9709029](#).
- [63] G. Curio and R. Y. Donagi, “Moduli in  $N = 1$  heterotic/F-theory duality,” *Nucl. Phys.* **B518** (1998) 603–631, [arXiv:hep-th/9801057](#).
- [64] J. Marsano, N. Saulina, and S. Schafer-Nameki, “A Note on G-Fluxes for F-theory Model Building,” *JHEP* **11** (2010) 088, [arXiv:1006.0483 \[hep-th\]](#).
- [65] J. Marsano, N. Saulina, and S. Schafer-Nameki, “On G-flux, M5 instantons, and U(1)s in F-theory,” [arXiv:1107.1718 \[hep-th\]](#).
- [66] T. W. Grimm, M. Kerstan, E. Palti, and T. Weigand, “Massive Abelian Gauge Symmetries and Fluxes in F-theory,” [arXiv:1107.3842 \[hep-th\]](#).
- [67] T. W. Grimm and T. Weigand, “On Abelian Gauge Symmetries and Proton Decay in Global F- theory GUTs,” *Phys. Rev.* **D82** (2010) 086009, [arXiv:1006.0226 \[hep-th\]](#).
- [68] A. Sen, “F theory and the Gimon-Polchinski orientifold,” *Nucl.Phys.* **B498** (1997) 135–155, [arXiv:hep-th/9702061 \[hep-th\]](#).
- [69] A. Sen, “Orientifold limit of F-theory vacua,” *Phys. Rev.* **D55** (1997) 7345–7349, [arXiv:hep-th/9702165](#).
- [70] A. Collinucci, F. Denef, and M. Esole, “D-brane Deconstructions in IIB Orientifolds,” *JHEP* **02** (2009) 005, [arXiv:0805.1573 \[hep-th\]](#).
- [71] A. P. Braun, R. Ebert, A. Hebecker, and R. Valandro, “Weierstrass meets Enriques,” *JHEP* **1002** (2010) 077, [arXiv:0907.2691 \[hep-th\]](#).
- [72] G. Aldazabal, A. Font, L. E. Ibanez, and A. M. Uranga, “New branches of string compactifications and their F- theory duals,” *Nucl. Phys.* **B492** (1997) 119–151, [arXiv:hep-th/9607121](#).
- [73] A. Klemm, P. Mayr, and C. Vafa, “BPS states of exceptional non-critical strings,” [arXiv:hep-th/9607139](#).
- [74] A. Klemm, B. Lian, S. S. Roan, and S.-T. Yau, “Calabi-Yau fourfolds for M- and F-theory compactifications,” *Nucl. Phys.* **B518** (1998) 515–574, [arXiv:hep-th/9701023](#).

- [75] P. Candelas, E. Perevalov, and G. Rajesh, “Comments on A, B, C chains of heterotic and type II vacua,” *Nucl. Phys.* **B502** (1997) 594–612, [arXiv:hep-th/9703148](#).
- [76] M. Bershadsky, T. Pantev, and V. Sadov, “F-theory with quantized fluxes,” *Adv. Theor. Math. Phys.* **3** (1999) 727–773, [arXiv:hep-th/9805056](#).
- [77] P. Berglund, A. Klemm, P. Mayr, and S. Theisen, “On type IIB vacua with varying coupling constant,” *Nucl. Phys.* **B558** (1999) 178–204, [arXiv:hep-th/9805189](#).
- [78] D. Lust, P. Mayr, S. Reffert, and S. Stieberger, “F-theory flux, destabilization of orientifolds and soft terms on D7-branes,” *Nucl. Phys.* **B732** (2006) 243–290, [arXiv:hep-th/0501139](#).
- [79] A. Braun, A. Hebecker, and H. Triendl, “D7-Brane Motion from M-Theory Cycles and Obstructions in the Weak Coupling Limit,” *Nucl. Phys.* **B800** (2008) 298–329, [arXiv:0801.2163 \[hep-th\]](#).
- [80] B. R. Greene, D. R. Morrison, and M. R. Plesser, “Mirror manifolds in higher dimension,” *Commun. Math. Phys.* **173** (1995) 559–598, [arXiv:hep-th/9402119](#).
- [81] A. Strominger, “SPECIAL GEOMETRY,” *Commun. Math. Phys.* **133** (1990) 163–180.
- [82] M. Esole and S.-T. Yau, “Small resolutions of SU(5)-models in F-theory,” [arXiv:1107.0733 \[hep-th\]](#).
- [83] A. Collinucci and R. Savelli, “On Flux Quantization in F-Theory,” [arXiv:1011.6388 \[hep-th\]](#).
- [84] A. Collinucci, “New F-theory lifts,” *JHEP* **08** (2009) 076, [arXiv:0812.0175 \[hep-th\]](#).
- [85] A. Collinucci, “New F-theory lifts II: Permutation orientifolds and enhanced singularities,” *JHEP* **04** (2010) 076, [arXiv:0906.0003 \[hep-th\]](#).
- [86] I. Brunner and K. Hori, “Orientifolds and mirror symmetry,” *JHEP* **11** (2004) 005, [arXiv:hep-th/0303135](#).
- [87] L. Martucci, “D-branes on general  $N = 1$  backgrounds: Superpotentials and D-terms,” *JHEP* **06** (2006) 033, [arXiv:hep-th/0602129](#).
- [88] L. Martucci and P. Smyth, “Supersymmetric D-branes and calibrations on general  $N = 1$  backgrounds,” *JHEP* **11** (2005) 048, [arXiv:hep-th/0507099](#).
- [89] J. Harris, “Algebraic Geometry: A first course,” *Springer Verlag* (1992) .
- [90] A. Beauville, “Determinantal hypersurfaces,” *Michigan Math J. Volume 48, Issue 1 (2000)*, 39-64.
- [91] P. Achinger, “Frobenius Push-Forwards on Quadrics,” [arXiv: 1005.0594v1 \[math\]](#).
- [92] N. Addington, “Spinor sheaves on singular quadrics,” [arXiv:0904.1766 \[math\]](#).
- [93] A. Kapustin and Y. Li, “D-Branes in Landau-Ginzburg Models and Algebraic Geometry,” *JHEP* **12** (2003) 005, [arXiv:hep-th/0210296](#).

- [94] I. Brunner, M. Herbst, W. Lerche, and B. Scheuner, “Landau-Ginzburg realization of open string TFT,” *JHEP* **11** (2006) 043, [arXiv:hep-th/0305133](#).
- [95] M. Baumgartl, I. Brunner, and M. R. Gaberdiel, “D-brane superpotentials and RG flows on the quintic,” *JHEP* **07** (2007) 061, [arXiv:0704.2666 \[hep-th\]](#).
- [96] M. Herbst, K. Hori, and D. Page, “Phases Of N=2 Theories In 1+1 Dimensions With Boundary,” [arXiv:0803.2045 \[hep-th\]](#).
- [97] I. Brunner, M. R. Gaberdiel, and C. A. Keller, “Matrix factorisations and D-branes on K3,” *JHEP* **06** (2006) 015, [arXiv:hep-th/0603196](#).
- [98] D.-E. Diaconescu, G. W. Moore, and E. Witten, “E(8) gauge theory, and a derivation of K-theory from M- theory,” *Adv. Theor. Math. Phys.* **6** (2003) 1031–1134, [arXiv:hep-th/0005090](#).
- [99] M. Alim, M. Hecht, P. Mayr, and A. Mertens, “Mirror Symmetry for Toric Branes on Compact Hypersurfaces,” *JHEP* **09** (2009) 126, [arXiv:0901.2937 \[hep-th\]](#).
- [100] M. Alim *et al.*, “Hints for Off-Shell Mirror Symmetry in type II/F-theory Compactifications,” *Nucl. Phys.* **B841** (2010) 303–338, [arXiv:0909.1842 \[hep-th\]](#).
- [101] T. W. Grimm, T.-W. Ha, A. Klemm, and D. Klevers, “Computing Brane and Flux Superpotentials in F-theory Compactifications,” *JHEP* **04** (2010) 015, [arXiv:0909.2025 \[hep-th\]](#).
- [102] M. Aganagic and C. Beem, “The Geometry of D-Brane Superpotentials,” [arXiv:0909.2245 \[hep-th\]](#).
- [103] T. W. Grimm, T.-W. Ha, A. Klemm, and D. Klevers, “Five-Brane Superpotentials and Heterotic/F-theory Duality,” *Nucl. Phys.* **B838** (2010) 458–491, [arXiv:0912.3250 \[hep-th\]](#).
- [104] H. Jockers, P. Mayr, and J. Walcher, “On N=1 4d Effective Couplings for F-theory and Heterotic Vacua,” [arXiv:0912.3265 \[hep-th\]](#).
- [105] M. Alim *et al.*, “Type II/F-theory Superpotentials with Several Deformations and N=1 Mirror Symmetry,” *JHEP* **06** (2011) 103, [arXiv:1010.0977 \[hep-th\]](#).
- [106] Dodson, C. T. J. and Parker, Phillip E., *A user’s guide to algebraic topology*, vol. 387 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1997.